ESSAYS ON MARITIME LOGISTICS MANAGEMENT

by

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This is to certify that I have examined the above PhD thesis
and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by
the thesis examination committee have been made.

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ABSTRACT

Maritime logistics is one essential part of the global supply chain, enabling the globalization of economy. It is of practical importance for improving the efficiency and reliability of maritime logistics at multiple levels. This thesis studies two operational issues in maritime logistics.

The first issue is about the concern on piracy attack, a serious security threat causing the affected shipping routes more costly and less reliable. Piracy attack occurs in many areas beyond the well-reported Somalia Pirates. By now, various strategic actions have been taken to prevent piracy attacks, such as rerouting vessels to avoid the dangerous water area, forming group transit and strengthening the navy patrols. However, these actions still are not enough to eliminate the possibility of piracy attacks. Therefore, it is important for a commercial vessel to be equipped with operational solutions in case of piracy attacks. In particular, choosing a direction for quickly running away is a critical real-time decision for the vessel.

This thesis starts analyzing a situation where a commercial vessel finds itself being chased by one pirate skiff. The vessel wants to make a good decision to evade the chasing. We formulate such an evading problem as a nonlinear optimal control problem. We consider different policies such as maintaining a straight direction and making turns. We start with the direct heading policy where the vessel will maintain its direction, and derive the condition under which such a policy is feasible for the vessel to be safe. Then we extend to the policy in which the vessel will make turns to evade the chasing. The feasibility condition

of these policies are derived, and we develop algorithms to optimize the policies under the concept of Pareto-optimal policy.

Based on the above result, we extend our research to study the situation with multiple pirate skiffs chasing one commercial vessel. The situation will become more complicated. For example, there exists a most conservative one-turn policy against one skiff, but that is not the case when there are two skiffs. Still, we are able to show that the policies derived against one skiff can be modified for the more challenging problem.

The second issue is about planning containers transportation in feeder lines. We consider a space allocation problem for a feeder vessel to collect/ deliver containers along its route. A feeder vessel departs from a hub port, sequentially calling for a number of ports to make container collection and delivery. There are two challenges in making the decision. First, the capacity of the vessel is shared by two types of containers, laden ones collected during the route and empty ones to deliver to each port, where the collection consumes the capacity and the delivery releases the capacity. Second, the demand has also two types, some demand having reservation made in advance but subject to random cancellation, some demand coming purely from random spot market. With the commitment of fulfilling realized demand with reservation, the vessel has to decide the fulfillment level to the demand on the spot, so as to maximize the expected revenue of the whole trip. We formulate the problem as a Markov decision process, and derive a two-dimensional threshold policy for serving the demands based on the concept of discretely concavity.

The technical contribution of the thesis lies in the application of optimization. It involves two different streams of optimization, nonlinear deterministic optimization and discrete stochastic optimization, both being hard optimization problems. We are able to successfully solve these problems with useful structural results revealed.

CHAPTER I

INTRODUCTION

1.1 Research Background

With the globalization, maritime logistics is playing a more and more important role in the global economy. Except for the huge opportunities, the globalization of the maritime logistics also bring some new challenges.

The first challenge comes with the maritime security for the long-haul line. Researchers have discussed the maritime safety and security problem from different perspectives, such as the humanity dimension and the shipping operational factors. Moreover, Chang et al.[1] undertake a survey to analyze the risk factors in container shipping based on the case of Taiwan. They identified 35 risk factors that may lead to maritime safety and security related damage from the literature review and advice from the container shipping industry. Before conducting the empirical study, those 35 factors are divided into 3 groups according to logistic flows in shipping operations, i.e., information flow, physical flow and finance flow. The result shows that the risks associated with physical flow would lead to a higher risk consequence, compared with the other two factors. One of the risk factor associated with physical flow is the attacks from the pirates, which should be paid more attention to.

The most famous pirate in modern time is the Somalia pirates. Due to the enormous efforts from the governments, international organizations and shipowners, the number of piracy attacks in Somalia is reduced significantly. However, the situation still remains chaotic. The International Chamber of Commerce (ICC) International Maritime Bureau's(IMB) annual piracy report reveals that piracy and armed robbery on the world's seas is persisting at levels close to those in 2014, despite reductions in the number of ships hijacked and crew captured. In somewhere, like South East Asia, the number of piracy attacks continue to rise. Reported by the International Maritime Organization(IMO), Southeast Asia now accounts for 60% of all incidents. The report also notes that the cyber risks in the maritime and shipping industry, which enables the pirates to identify target cargoes and obtain about more vulnerable ships and locations, require industry attention.

In [2], Helmick discussed some feature of modern maritime piracy and the major impacts and costs of piracy attacks on the global supply chain operations. He also discussed some strategies that could be taken to avoid and deter the pirates, such as implementing suggestions from BMP [3], enhancing antipiracy training, seeking military intervention, forming corridors and group transits during piracy water area, and rerouting to avoid the dangerous trade lanes and port zone. However, all of these high-level strategies will induce huge cost to the global supply chains. An infographic released by Nature's Water states that the annual financial loss around the globe due to the piracy attack is about US$13-16 billion, with 75% of piracy incidents occurring in Aisa. In case of an accidental encounter with the piracy attack, the vessel need be equipped with some operational solutions. In Chapter 2, we consider an antipiracy problem from the perspective of operational level, where a commercial vessel tries to get rid of the chasing from a pirate skiff. A simplest way to evade the chasing is to choose the direction same as the pirate, which will result in the biggest distance between the commercial vessel and the pirate skiff. However, the vessel may move backward its original direction, which is not preferred. Hence, we consider some other policies such that the vessel could arrive at a preferred position. Another policy is to maintain its direction, called direct heading policy. We derive the feasibility condition for this policy and conduct computational experiments on the Infeasible Region that the commercial vessel need detect the suspected piracy activities. We also propose the one-turn and two-turn policy when the direct heading policy is infeasible and characterize the optimal turn policy. In general, there might be multiple skiffs chasing the commercial vessel, used to be 2 skiffs [2]. Hence, we extend our results in Chapter 2 to the situation with two skiffs and discuss how to find the optimal turn policy based on the result with only one skiff.

Another issue is about the service for the feeder lines. Generally, two regional ports in the global maritime logistics system are not connected directly. Commodities and products from the exported port are first transported to the corresponding hub port, and then transported to other hub ports. After that, they will be distributed to their destination by feeder vessels.

Feeder lines have provided for important transport connection between the regions and the mainstream intercontinental lines. Different issues related to feeder lines could be addressed from the perspective of different planning levels. From the perspective of strategic level and tactical level, people are interested in the feeder line network design, including the selection of hub port, allocating the regional ports, determining the calling sequence of the regional ports and so on. Meanwhile, operational plannings in the feeder lines mainly focus on the management of the feeder vessel, like optimal service speed, shipping capacity utilization and amendment of routes. What we are interested in here is about the shipping capacity utilization of the feeder vessel to enhance the efficiency of the maritime transportation.

In practice, a feeder vessel is designed to serve the regional ports on a regular basis according to a fixed schedule. Generally, the vessel will take in-advance reservation before it starts sailing, and in common sense, the reservation can be canceled without penalty. In Chapter 4, we investigate a space allocation problem for a feeder vessel to decide the number of laden containers to collect and the number of empty containers to deliver during its trip along a predefined route. Besides the demand with reservation, there is demand on spot market. The realized demand with reservation must be guaranteed while the demand on spot market could be rejected. The optimal serving policy is derived, which also helps to figure out the optimal shipping capacity of the feeder vessel to be deployed for this route.

1.2 Thesis Organization

The rest of the thesis is organized as follows. In Chapter 2, we consider a problem where a commercial vessel is chased by one pirate skiff. Different policies are considered for the vessel and the conditions under which the policies are effective and safe for the vessel. Algorithms are also developed to optimize the policies based on the concept of "Pareto optimal policy". In Chapter 3, we extend the result in the Chapter 2 to the situation where there are multiple pirate skiffs chasing the commercial vessel individually. We discuss how to find a feasible policy for the commercial vessel to evade all the skiffs, and to optimize the policy with some computational results. In Chapter 4, we investigate a space allocation problem where a feeder vessel needs to decide how to allocate its space by collecting a suitable number of laden containers and delivering a suitable number of empty containers during its trip and derive the optimal serving policy. We summarize our major contribution in Chapter 5, and preset the technical proofs in the Appendices.

CHAPTER II

EVADING POLICIES FOR A VESSEL BEING CHASED BY ONE PIRATE SKIFF

2.1 Introduction

Piracy attacking has been a constant threat to maritime transport for hundreds of years. In the past decade, piracy attacks have headlined the news again in the media, mainly due to the suddenly spreading attacks originating from the Somali area. With the tremendous efforts from the international society, the situation of Somali piracy seems to be much improved as of today. However, the severity of the piracy problem is actually much beyond what had happened around the Somali coastal. According to ICC Commercial Crime services (Table 2.1), there were 246 reported attacks worldwide in 2015, zero occurred in the Somali area.

Table 2.1: Number of actual and attempted piracy attacks, 2010 - 2015

Locations	2010	2011	2012	2013	2014	2015
South- East Asia	70	80	104	128	141	147
Indian Subcontinent	23	10	11	12	21	11
Americas	40	25	17	18	4	8
Somalia/Gulf of Aden	219	236	75	15	11	0
Nigeria	19	10	27	31	18	14
Other Africa	17	47	48	32	26	21
Rest of World	57	31	15	28	24	45
Total	445	439	297	264	245	246

(source: https://www.icc-ccs.org)

A modern pirate group often operates by using motherships and skiffs (high speed small boats). A mothership is a large slow ship carrying pirates, food, and fuel, enabling pirates to extend their territory to a much larger area than before. Attacking skiffs with a high speed are often towed behind the Motherships. Once identifying a commercial vessel as target, pirates take skiffs, usually one or two, to rush to the vessel. The chasing may result in gun fairs when the skiffs are close enough to the vessel, and pirates will start climbing up the vessel if the skiff can catch up with the vessel. Due to the limited fuel supply of a skiff, the pirates will stop chasing after certain time without getting close to the vessel. The chasing may last one to two hours.

To address the issue of piracy attacks, the Maritime Safety Committee of IMO made a series of recommendations to ocean carriers in the Best Management Practices ([3]). According to BMP, carriers should consider preventive actions such as avoiding certain high risk area, joining group transit schemes with military or independent convey, strengthening ship protection measures such as making the ship hard to be climbed, and so on. In practice, however, preventive actions cannot completely eliminate the risk of being attacked when a commercial vessel is sailing on the sea. If finding a pirate skiff approaching in real time, a commercial vessel can only run away, as suggested in BMP:

"one of the most effective ways to defeat a pirate attack is by using speed to try to outrun the attackers and/or make it difficult to board" and "try to steer a straight course to maintain a maximum speed."

Intuitively, running away at the maximum speed seems to be a straightforward action to take, so the problem is rather simple. However, there are indeed other factors to consider. For example, the direction towards which the vessel sails. Note that each vessel has a planned route to its destination. It would be the ideal case if the vessel can evade the chasing by speeding along the planned route, a policy referred to as direct heading hereafter. Unfortunately, direct heading is not always safe or feasible, which depends on the positions of the pirate skiffs. Therefore, the vessel needs to consider turning its sailing direction (alter course), making the vessel deviate from the planned route. The most conservative choice is changing to the direction exactly opposite to the chasing skiff, but that may deviate significantly from the planned route. After hours of evading at the maximum speed, the vessel may end up at a safe place dozens of nautical miles away from the planned route, which will cost the vessel additional fuel to return to the planned route. In fact it may not be necessary for the vessel to take such a conservative action. There should be a direction that makes the vessel safe and at the same time keeps the vessel close to the planned route as much as possible.

In this paper, we aim to study such an evading problem for a commercial vessel that is being chased by pirate skiffs. The purpose is to design a cost-effective and safe strategy by optimizing the steering direction of the vessel. We highlight our main results and contribution as follows.

We formally formulate the problem by identifying the key factors in the chasing process. We first present a dynamic differential game model with a pirate skiff as pursuer and the commercial vessel as evader. Assuming that the pirate skiff takes a greedy policy: the pure pursuit guidance law, we then transform the game into an optimal control problem for the vessel.

To solve the problem, we focus on three practical policies for the vessel, direct heading, making one turn, and making two turns. Note that such policies are consistent with the BMP recommendation because making too many turns will reduce the speed of the vessel. For each policy, we are able to derive a few structural properties, and develop algorithms to efficiently compute the optimal decisions.

We conduct extensive simulation to validate our policies. The results show that our policies can lead to safe and cost-effective decisions for the commercial vessels. Our model can generate a set of Pareto-optimal solutions, making it possible for the vessel to evaluate and make decisions in different scenarios.

The rest of the paper is organized as follows. In Section 2.2, we review the related literature. In Section 2.3, we introduce our model and problem formulation. Then in Sections 2.4 and 2.5, we study different policies when there is a single chasing skiff. We conclude the paper in Section 2.6.

2.2 Literature Review

Fighting the piracy attack is an old problem that has been discussed in different areas. For example, [4] and [5] study the activities of Somali piracy, [6] and [7] address the impact of piracy on the global economy as well as the transportation cost. More detailed literature review on maritime safety and security can be found in [8]. The work on quantitative operations models of the battle with piracy now attracts more attentions, but the result is still scarce. [9] uses simulation to evaluate the risk level of a specific water area, [10] uses simulation to compare a variety of prevention operations, and [11] studies how to improve the group transit schemes. All such works belong to proactive actions which are important to take in advance. However, proactive actions also means a huge cost to the supply chain. Our work is different in that we focus on reactive actions to be taken in real time.

From a boarder point of review, the pirate chasing problem that we study here belongs to the general pursuit-evasion problem that has been studied in other situations, especially in military applications such as the cruise missile attack. The related research can be classified into three categories: the pursuit problem, the evasion problem and more general pursuit-evasion game. In what follows, we give an overview of such problems. We will explain the uniqueness of our problem in the next section after we present the details of our problem.

The pursuit problem studies the pursuer's strategies to chase or intercept a moving target. We now present three fundamental guidance strategies, which is applicable for planar motion control, see [12], namely line of sight (LOS) law, pure pursuit (PP) law and proportional navigation (PN) law. These guidance strategies are referred to as the classical guidance laws. The principle of LOS guidance law, e.g. [13], is to guide the pursuer on a LOS course of a ground station and the evader. Under pure pursuit law, e.g., [13] and [14], the pursuer always aligns its velocity along the LOS angle between the evader and itself. If the pursuer has a higher speed than the evader, it is guaranteed that an intercept will occur if the chasing time is long enough. However, the miss-distance performance is not satisfactory, which is because the pure pursuit guidance law requires high latax towards the end of the engagement. Under the proportional navigation, e.g., [13], [15], [16] and [17], the pursuer just selects the rotation rate of its velocity directly proportional to the rotation of rate of the LOS angle between the pursuer and the evader. The proportional navigation law has a better performance than the pure pursuit law on the miss-distance. However, PN law does not perform well against the maneuvering evader, see [18] and [19]. It is because that it doesn't consider the acceleration of the evader, especially when the evader has a higher speed.

Modern guidance laws mainly based upon the optimal control theory, including the improved PN laws, predictive guidance and even the differential games. However, it is quit computationally intensive to implement these guidance laws. As the first step to study the pirate-chasing problem, we assume that the pirate adopts the pure pursuit law in this paper. We will given justification for this assumption when we formally define the problem.

The evasion problem, on the other hand, focuses on the evader's strategies to get rid of the chasing, e.g., [20], [21], [15] and [22]. In all these works, the strategy of the pursuer is predefined and the goal of the evader is to find the safest strategy by maximizing the capturing time or to find a cost-effective strategy by minimizing the fuel cost. The pursuit-evasion game studies the problem by simultaneously considering the pursuer's and the evader's strategies, e.g., [23], [24] and [25], [26], and [27] .

Generally the pursuit problem to induce modern guidance laws, the evasion problem and the pursuit-evasion problem are formulated as optimal control problem. There are two approaches in the literature, direct method and indirect method. With the direct method, the optimal control problem is reformulated into a nonlinear programming by the discretization method, and then solved with nonlinear programming techniques directly, e.g., [21], [24] and [25]. The indirect method focuses on identifying the optimality conditions first, e.g., [15] and [22]. In this paper, we follow the indirect method where we can obtain not only some algorithms to solve the problem, but also the certain general guidance that will be revealed by the optimality conditions.

2.3 Problem Formulation

We consider a pirate chasing situation, where a commercial vessel finds itself being chased by a pirate skiff at time zero. Consequently, the commercial vessel needs to decide its sailing policy, including its sailing direction and speed such that it can escape from the chasing; at the same time, the pirate skiff wants to catch the commercial vessel as early as possible. Hereafter we will use the vessel to refer to the commercial vessel, and the pirate to refer to the pirate skiff.

We can use a Cartesian coordinate system to describe the problem. As depicted in Figure 2.1, the initial position of the vessel is at the origin $(0, 0)$, and the initial sailing direction is towards the x-axis. Without loss of generality, assume that the initial position (x_0, y_0) of the pirate skiff is below or on the x-axis, i.e., $y_0 \leq 0$. At any time t, let $(x_v(t), y_v(t))$ and $(x_p(t), y_p(t))$ denote the positions of the vessel and the pirate, respectively. Let $v_v(t)$ denote the vessel's speed and $\alpha(t)$, an angle formed with the x-axis, denote its direction. Similarly, let $v_p(t)$ and $\beta(t)$ denote the sailing speed and direction of the pirate, respectively.

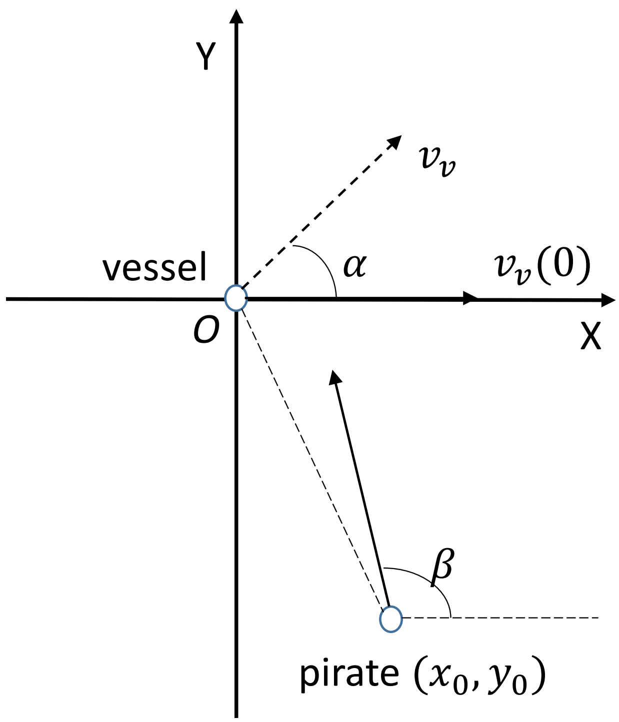


Figure 2.1: The initial situation

The dynamic process of the system can be described by the following kinematic equations,

$$\begin{cases} \frac{\mathrm{d}x_v(t)}{\mathrm{d}t} = v_v(t)\cos\alpha(t), \\ \frac{\mathrm{d}y_v(t)}{\mathrm{d}t} = v_v(t)\sin\alpha(t), \\ \frac{\mathrm{d}x_p(t)}{\mathrm{d}t} = v_p(t)\cos\beta(t), \\ \frac{\mathrm{d}y_p(t)}{\mathrm{d}t} = v_p(t)\sin\beta(t), \end{cases} \tag{2.1}$$

with the initial condition

$$\begin{cases} (x_v(t), y_v(t))|_{t=0} = (0,0), \\ (x_p(t), y_p(t))|_{t=0} = (x_0, y_0). \end{cases}$$

The above equations define a general pursuit-evasion game where both the vessel and the pirate have to determine their policies. In our problem, one key concern is the distance between them. Let

$$r(t) = \sqrt{(x_v(t) - x_p(t))^2 + (y_v(t) - y_p(t))^2} \tag{2.2}$$

denote their distance at time t, and $r_0 = \sqrt{x_0^2 + y_0^2}$ denote the initial distance. The following assumptions streamline the definition of our problem.

Assumption 1. *The vessel is safe if and only if $r(t) \geq R$ for $t \in [0, T]$, where R and T are given positive constants.*

The first constant R indicates a minimum safety distance. For example, R can be set at the longest distance at which the pirate may start to open fire. As long as the vessel keeps away from the pirate at a distance no less than R, the vessel can be assumed to be safe. To avoid the trivial case, we assume $r_0 \geq R$. The second constant T denotes the arriving time of the rescue or the maximum time before which the pirate gives up chasing the vessel due to the limited fuel on the skiff. For example, it is reported that the pirates will chase for up to 2 hours in many cases.

We first consider how the pirate may determine $\beta(t)$, the sailing direction of the skiff. We assume that the pirate will always dash directly towards the vessel, which is known as the pure pursuit guidance law. Specifically, at any time t, we use $\theta(t)$ to denote the angle of the pirate's LOS towards the vessel with respect to the x-axis, as shown in Figure 2.2. Then the pirate will set the sailing direction $\beta(t)$ at $\theta(t)$. Note that, by definition, for $r(t) > 0$ we have

$$\cos\theta(t) = \frac{x_v(t) - x_p(t)}{r(t)}, \text{ and } \sin\theta(t) = \frac{y_v(t) - y_p(t)}{r(t)}. \tag{2.3}$$

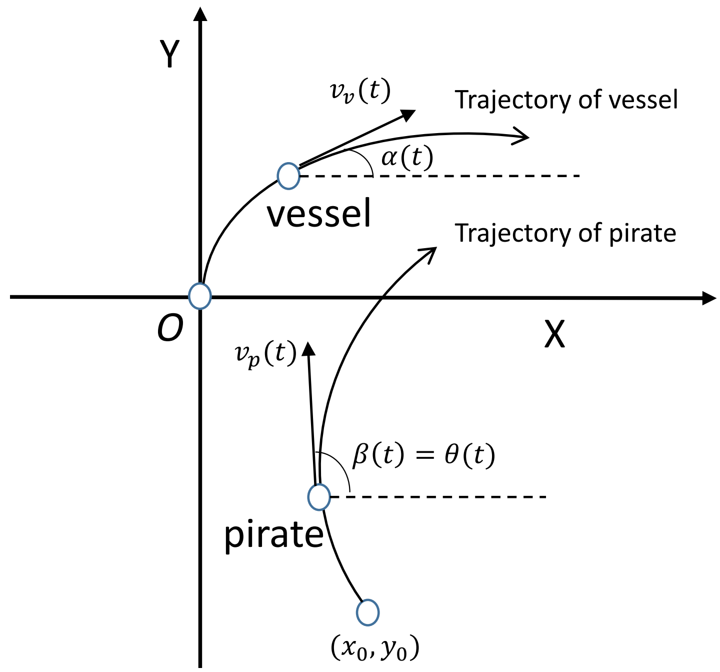


Figure 2.2: Dynamic process under the pure pursuit guidance law

Assumption 2. *The pirate will take the pure pursuit guidance law in which* $\beta(t) = \theta(t)$ *for* $t \in [0, T]$.

Assuming that the pirate wants to catch the vessel as early as possible, then we can partially justify the pure pursuit guidance law by the following lemma. The lemma implies that taking the pure pursuit guidance law is a locally optimal policy for the pirate because it leads to the steepest descent of $r(t)$ at any time t. Note that Eq.(2.3) is given under the condition $r(t) > 0$, which is the meaningful case for our problem. In what follows, our analysis is also applied to the case of $r(t) > 0$ unless otherwise specified, though we will denote the time range as $t \in [0, T]$. For simplicity, we will not state the condition $r(t) > 0$ again.

Lemma 2.1. *At any time* t, $\dfrac{\mathrm{d}r(t)}{\mathrm{d}t}$ *is minimized by* $\beta(t) = \theta(t)$.

Next, we consider the speed decisions of both the pirate and the vessel.

Assumption 3. *The speeds of both the vessel and the pirate are constant, i.e.,* $v_v(t) \equiv v_v$, $v_p(t) \equiv v_p$, *for* $t \in [0, T]$.

This assumption is consistent with some practical evidence. In particular, BMP suggests that a vessel steers a straight course to maintain a maximum speed. At the same time, the pirate also has a reasonable incentive to use the maximum speed in order to catch the vessel as soon as possible, for example, before any rescue team arrives. Thus we consider the common case where both the pirate and vessel take their respective maximum speed. Actually our analysis can also be used for the vessel to evaluate the option of taking a slower speed as long as the speed does not vary over time.

For notional convenience, let $\gamma = \frac{v_p}{v_v}$ denote the ratio of speeds between the pirate and the vessel. Note that $\gamma > 1$ means that the pirate has a higher speed over the vessel, the more often case resulting in a successful hijack. When $\gamma = 1$, i.e., the pirate and vessel have the same speed, we then use v to denote the speed.

Given the above assumptions, we can simplify the system formulation (2.1) by replacing $v_v(t)$ by v_v, $v_p(t)$ by v_p, and $\beta(t)$ by $\theta(t)$. In addition, for the position of the pirate, we use $r(t)$ and $\theta(t)$ to characterize its relative position to the vessel, thus eliminating the notion of $x_p(t)$ and $y_p(t)$. Now consider the dynamic process of $\theta(t)$. Differentiating on both sides of (2.3), we have

$$-\sin\theta(t)\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \frac{\frac{\mathrm{d}x_v(t)}{\mathrm{d}t} - \frac{\mathrm{d}x_p(t)}{\mathrm{d}t}}{r(t)} - \frac{(x_v(t) - x_p(t))\frac{\mathrm{d}r(t)}{\mathrm{d}t}}{r^2(t)}.$$

Substituting (2.1) and (2.3) to the above equation, we have

$$\begin{aligned} -\sin\theta(t)\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} &= \frac{v_v\cos\alpha(t) - v_p\cos\theta(t) - (v_v\cos(\alpha(t)-\theta(t)) - v_p)\cos\theta(t)}{r(t)} \\ &= -\frac{v_v\sin(\alpha(t)-\theta(t))\sin\theta(t)}{r(t)}, \end{aligned}$$

and thus

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \frac{v_v\sin(\alpha(t)-\theta(t))}{r(t)}. \quad (2.4)$$

Based on the above results, the system can be reformulated as follows,

$$\begin{cases} \frac{\mathrm{d}x_v(t)}{\mathrm{d}t} = v_v\cos\alpha(t), \\ \frac{\mathrm{d}y_v(t)}{\mathrm{d}t} = v_v\sin\alpha(t), \\ \frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_v(\cos(\alpha(t)-\theta(t)) - \gamma), \\ \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = v_v\frac{\sin(\alpha(t)-\theta(t))}{r(t)}, \end{cases} \quad (2.5)$$

with the safety constraint:

$$r(t) \geq R\ , \quad \text{for} \quad t \in [0, T]. \tag{2.6}$$

In the new formulation, the only decision is $\alpha(t)$, the sailing direction of the vessel. The problem becomes an optimal control problem where a feasible control policy $\alpha(t), t \in [0, T]$ is needed to optimize some objective function.

While any policy $\alpha(t)$ satisfying (2.5) and (2.6) is feasible, the vessel may choose one that is also cost-effective. For example, the vessel may simply choose $\alpha(t) = \theta(t)$, sailing just opposite to the pirate, but that may lead the vessel to a position very far away from the original route, and after the chasing, the vessel needs to sail additional voyage to return. A more reasonable policy for the vessel should be ending up at the position close to the original route. To this end, we consider the end position $(x_v(T), y_v(T))$, where $x_v(T)$ measures the movement along the original direction, and $y_v(T)$ gives the deviated distance. In general, we hope $x(T)$ to be as large as possible, and $y_v(T)$ to be close to zero as much as possible.

Definition 2.1. *For any two feasible policies $\alpha(t)$ and $\alpha'(t)$ on $t \in [0, T]$, we say $\alpha(t)$ dominates $\alpha'(t)$ if and only if one of the following two statements is true:*

1) $x_v(T) > x'_v(T)$ for the case of $x'_v(T) < 0$, or

2) $x_v(T) \geq x'_v(T)$ and $|y_v(T)| \leq |y'_v(T)|$ for the case of $x'_v(T) \geq 0$, and at least one inequality is not tight.

The meaning of dominance can be explained as follows. The common condition $x_v(T) \geq x'_v(T)$ implies that under $\alpha(t)$ the vessel moves longer distance along the planned direction, which is preferred. For the case $x'_v(T) > 0$, we may also prefer a smaller deviation from the planned direction, so we add another condition on $y_v(T)$. For the case $x'_v(T) < 0$, it is reasonable to give priority to shorter backward sailing, so the condition on $y_v(T)$ is not added. By definition, a vessel does not need to consider a policy $\alpha'(t)$ if it is dominated by another policy $\alpha(t)$.

Definition 2.2. *A control policy $\alpha(t)$, $t \in [0, T]$, is Pareto-optimal if and only if $\alpha(t)$, $t \in [0, T]$, is not dominated by any other control policy. If all Pareto-optimal controls have the same end position $(x_v(T), y_v(T))$, then they are global optimal.*

Any Pareto-optimal policy has certain advantage against another one, with respect to either sailing a longer distance along the x-axis or having a smaller deviation along the y-axis. In this paper, our goal is to characterize the frontier of the set of the Pareto-optimal policies. This enables the vessel master to understand and evaluate all possible choices for making a decision.

2.4 Direct Heading

Direct heading refers to the policy under which the vessel keeps the original sailing direction, i.e. $\alpha(t) \equiv 0$ for $t \in [0, T]$, leading to $(x_v(T), y_v(T)) = (v_v T, 0)$. This means that direct heading will be the unique global optimal policy if it satisfies (2.5) and (2.6) because $x_v(T)$ achieves its maximum value and $|y_v(T)| = 0$ is at its minimum value. So the issue related to direct heading is testing its feasibility, which is determined by the initial position of the pirate (r_0, θ_0).

2.4.1 Feasibility Test for Direct Heading

The feasibility test is to check if $r(t) \geq R$ for $t \in [0, T]$. We first discuss two special cases with respect to θ_0, $\theta_0 = 0$ and $\theta_0 = \pi$. For the case of $\theta_0 = 0$, i.e., the pirate appears directly behind the vessel, the pirate will also keep its direction as $\theta(t) = 0$, so direct heading is feasible as long as $\gamma \leq 1$, or $\gamma > 1$ but $(v_p - v_v)T \leq r_0 - R$. For the case of $\theta_0 = \pi$, i.e., the pirate appears directly in front of the vessel, direct heading means that the vessel sails directly towards to the pirate, which is not a reasonable choice at all. In the following analysis, we will exclude these two special cases and assume $\theta_0 \in (0, \pi)$, i.e., the pirate does not appear exactly on the vessel's sailing direction.

To facilitate the discussion, we first introduce a definition.

Definition 2.3. *The capture time, denoted by T_c, is the earliest time when the distance $r(t)$ between the vessel and the pirate decreases to R, i.e., $r(t) > R$ for $t \in [0, T_c)$ and $r(T_c) = R$.*

Note that T_c may or may not exist. If T_c does not exist, the vessel will always be safe regardless of the chasing time limit T. If T_c exists and $T_c > T$, the vessel is still safe before

T. So the feasibility test can be done by checking the existence of T_c and calculating its value if existing.

Under the direct heading policy where $\alpha(t) \equiv 0$, the processes of $r(t)$ and $\theta(t)$, before $r(t) = 0$, can be simplified into

$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_v(\cos\theta(t) - \gamma), \tag{2.7}$$

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = -v_v\frac{\sin\theta(t)}{r(t)}. \tag{2.8}$$

An analytical solution of $r(t)$ on $\theta(t)$ can be found in [13]. Specially, when $\theta(t)$ is known to be in $(0, \pi)$, the solution is given as

$$r(t) = C_0\frac{\tan^{\gamma}\frac{\theta(t)}{2}}{\sin\theta(t)}, \tag{2.9}$$

and $\theta(t)$ satisfies

$$-\frac{1}{2(1+\cos\theta(t))} + \frac{1}{4}\ln\Big(\frac{1+\cos\theta(t)}{1-\cos\theta(t)}\Big) = \frac{v}{C_0}t + C_1, \text{if } \gamma = 1, \tag{2.10}$$

$$\frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta(t)}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta(t)}{2} = -\frac{v_v}{C_0}t + C_2, \text{if } \gamma \neq 1, \tag{2.11}$$

where $C_0 = \frac{r_0\sin\theta_0}{\tan^{\gamma}\frac{\theta_0}{2}}$, $C_1 = -\frac{1}{2(1+\cos\theta_0)} + \frac{1}{4}\ln(\frac{1+\cos\theta_0}{1-\cos\theta_0})$, and $C_2 = \frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta_0}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta_0}{2}$ are three parameters depending on the initial position of the pirate (r_0, θ_0) and the speed ratio of the two objects γ.

To make the solution (2.9)-(2.11) valid to our problem, we need to guarantee $\theta(t) \in (0, \pi)$ for $t \in [0, T_c)$. From (2.8), we can see that $\theta(t)$ is non-increasing on t for $\theta(t) \in [0, \pi]$ because $\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \leq 0$. This means that, under the initial condition $\theta_0 \in (0, \pi)$, $\theta(t)$ is strictly decreasing until possibly at a certain time $t = \tau$, we have $\theta(\tau) = 0$, and then $\theta(t) = 0$ for $t > \tau$. The next lemma discusses the existence of such a τ.

Lemma 2.2. *If $\gamma > 1$, there exists a time τ such that $\theta(\tau) = 0$, $\theta(t)$ is strictly decreasing on t and $\theta(t) > 0$ for $t \in [0, \tau)$, and furthermore, we have $r(t) > 0$ for $t \in [0, \tau)$ and $r(\tau) = 0$. If $\gamma \leq 1$, such a time τ does not exist, i.e., $\theta(t) > 0$ for all $t \in [0, +\infty)$.*

Lemma 2.2 implies that the solution given in (2.9)-(2.11) fully characterizes the process that we are interested in. Based on that, we can do the feasibility test by checking the existence of the capture time T_c. By definition, if T_c exists, it should satisfy $r(T_c) = R$ together with (2.10) for the case of $\gamma = 1$, and satisfy $r(T_c) = R$ together with (2.11) for the case of $\gamma \neq 1$.

We first consider the case of $\gamma = 1$, i.e., the pirate and the vessel have the same speed. In this case, an analytical solution is available. It gives the necessary and sufficient condition for the direct heading policy to be safe to the vessel.

Proposition 2.1. *When $\gamma = 1$, if $r_0(1 + \cos\theta_0) \geq 2R$, the capture time T_c does not exist, and if $r_0(1 + \cos\theta_0) < 2R$,*

$$T_c = \frac{r_0 - R}{2v_v} + \frac{r_0(1 + \cos\theta_0)}{4v_v} \ln \frac{r_0(1 - \cos\theta_0)}{2R - r_0(1 + \cos\theta_0)}.$$

Next, we consider the case of $\gamma \neq 1$. In this case we cannot obtain a closed-form solution for the capture time T_c. So we need to computationally find T_c, which is the solution of the following nonlinear equations of t.

$$\begin{aligned} &C_0 \frac{\tan^\gamma \frac{\theta(t)}{2}}{\sin\theta(t)} = R \\ &\frac{1}{2(\gamma - 1)} \tan^{\gamma-1} \frac{\theta(t)}{2} + \frac{1}{2(\gamma + 1)} \tan^{\gamma+1} \frac{\theta(t)}{2} = -\frac{v_v}{C_0} t + C_2 \end{aligned} \tag{2.12}$$

Before presenting our algorithm to solve (2.12), we first discuss the existence, uniqueness, and possible range of the solution. First, the case is simple if the pirate skiff has a higher speed, i.e., $\gamma > 1$. The next proposition gives a finite upper bound of T_c.

Proposition 2.2. *When $\gamma > 1$, (2.12) has a unique solution T_c where $T_c \in (0, \frac{r_0(\gamma + \cos\theta_0)}{v_v(\gamma^2 - 1)})$.*

The proof of Proposition 2.2 relies on a key fact that the distance function $r(t)$ is decreasing on t until $r(t) = 0$. When $\gamma < 1$, i.e., the pirate has a lower speed, the distance function $r(t)$ may not be monotone on t. It may be decreasing first, then become increasing. So the situation is more complicated. To characterize the shape of $r(t)$, we define three critical parameters that can be calculated directly for any given r_0, θ_0, and γ.

$$\begin{aligned} &\bar{\theta}(\gamma) \triangleq \arccos\gamma, \text{ where } \bar{\theta}(\gamma) \in (0, \pi/2), \\ &\bar{r}(\gamma) = C_0 \frac{\tan^\gamma \frac{\bar{\theta}(\gamma)}{2}}{\sin\bar{\theta}(\gamma)}, \\ &\bar{t}(\gamma) = \frac{r_0(\gamma + \cos\theta_0) - 2\gamma\bar{r}(\gamma)}{v_v(\gamma^2 - 1)}. \end{aligned}$$

The first parameter $\bar{\theta}(\gamma)$ gives a threshold value on θ_0 for the existence of T_c. Specifically, when $\theta_0 \leq \bar{\theta}(\gamma)$, T_c does not exist for any $r_0 > R$. The second parameter $\bar{r}(\gamma)$ is the minimum value of the distance function $r(t)$, giving a sufficient and necessary condition for the existence of T_c when $\theta_0 > \bar{\theta}(\gamma)$. The third parameter $\bar{t}(\gamma)$ is an upper bound of T_c when T_c does exist. In fact, $r(\bar{t}(\gamma)) = \bar{r}(\gamma)$. These are formally given in the following proposition.

Proposition 2.3. *When $\gamma < 1$,*

1) *$\bar{r}(\gamma)$ is the global minimum value of $r(t)$,*

2) *if $\theta_0 \leq \bar{\theta}(\gamma)$, or $\theta_0 > \bar{\theta}(\gamma)$ but $\bar{r}(\gamma) > R$, T_c does not exist, and*

3) *if $\theta_0 \leq \bar{\theta}(\gamma)$ and $\bar{r}(\gamma) \leq R$, T_c exists in $(0, \bar{t}(\gamma)]$.*

From Propositions 2.2 and 2.3, we know that, when $\gamma > 1$, T_c is the unique solution to (2.12) within the interval $t \in (0, \frac{r_0(\gamma+\cos\theta_0)}{\gamma^2-1})$, and when $\gamma < 1$, T_c is the unique solution to (2.12) within the interval $t \in (0, \bar{t}(\gamma)]$.

Directly solving (2.12) is hard; even with a given t, there is no closed-form equation to calculate $r(t)$. However, we notice that, for a given $\theta(t)$, it is easy to calculate both $r(t)$ and t. Because of the one-to-one mapping between t and $\theta(t)$ within the concerned intervals, in particular, $t \in (0, \frac{r_0(\gamma+\cos\theta_0)}{\gamma^2-1})$ corresponding to $\theta(t) \in (0, \theta_0)$ for the case $\gamma > 1$, and $t \in (0, \bar{t}(\gamma)]$ corresponding to $\theta(t) \in [\theta_\gamma, \theta_0)$ for the case $\gamma < 1$ and T_c exits, we can solve (2.12) by the following algorithm.

Algorithm 1.

Step 1. Solve $C_0 \frac{\tan^\gamma \frac{\theta(t)}{2}}{\sin\theta(t)} = R$ with respect to $\theta(t)$, for $\theta(t) \in (0, \theta_0)$ when $\gamma > 1$, and for $\theta(t) \in [\bar{\theta}(\gamma), \theta_0)$ when $\gamma < 1$. Let the solution be θ.

Step 2. Given the solution θ found in Step 1, calculate t from (2.11) with $\theta(t) = \theta$. Then $T_c = t$.

Step 1 is to solve a standard unconstrained nonlinear problem, which can be done either by a bisection search or Newton's method. Converging to the unique solution is guaranteed by Propositions 2.2 and 2.3.

2.4.2 Infeasible Region

Given any initial position of the pirate, (r_0, θ_0), we are able to use Propositions 2.1 to 2.3 to check the feasibility of direct heading. Based on that, we can further characterize the entire set of initial positions under which directing heading is infeasible, where we refer the set as *Infeasible Region*. It is helpful to construct the Infeasible Region in advance. For example, the vessel should strengthen the surveillance over the Infeasible Region. In addition, when a pirate skiff is found with an uncertain speed, the vessel can evaluate a number of scenarios by checking the Infeasible Regions with different parameters, which enables the vessel to make a decision in real time.

We first study how the Infeasible Region depends on the initial position (r_0, θ_0). To this end, we slightly modify the notation $r(t)$ to $r(t, r_0, \theta_0)$, and $\bar{r}(\gamma)$ to $\bar{r}(\gamma, r_0, \theta_0)$, in order to explicitly show its dependence on (r_0, θ_0). From the previous analysis, we know that the Infeasible Region is given by $\{(r_0, \theta_0) : r(T, r_0, \theta_0) < R\}$ when $\gamma \geq 1$, and given by $\{(r_0, \theta_0) : r(T, r_0, \theta_0) < R \text{ if } T < \bar{t}(\gamma), \text{ and } \bar{r}(\gamma, r_0, \theta_0) < R \text{ if } T \geq \bar{t}(\gamma))$ when $\gamma < 1$.

Proposition 2.4. *For any given chasing time T and speed ratio γ, if direct heading policy is feasible when the initial position of the pirate is (r_0, θ_0), then it is feasible for the position being (r, θ_0) where $r \geq r_0$ and (r_0, θ) where $\theta \leq \theta_0$.*

Proposition 2.4 shows that when the pirate is relatively behind the vessel's sailing direction with the same distance r_0 or is far away from the vessel along the same direction θ_0, the vessel is more likely to be safe by simply sailing directly. This proposition also implies that the Infeasible Region is a compact set with respect to the initial position (r_0, θ_0). Thus, to construct the Infeasible Region, it is equivalent to find the border of the region. All positions insider the border will lead direct heading policy to infeasible while the positions outside will guarantee the safety of the commercial vessel.

To find the border efficiently, we propose the algorithms as follows. The following bisection method is to find the unique relative distance r_0 such that $C_0 \frac{\tan^{\gamma} \frac{\theta(T)}{2}}{\sin \theta(T)} = R$ for given θ_0.

Algorithm 2.

- Initialization: $\theta(T)_u = \theta_0$, $\theta(T)_l = 0$;
- If $\theta_0 \le \bar{\theta}(\gamma)$, $r_0 = R$. Otherwise, solve equation (2.10) when $\gamma = 1$ and equation (2.11) when $\gamma \neq 1$ to obtain r_0 for given $\theta(T) = \frac{\theta(T)_l + \theta(T)_u}{2}$;
- Calculate $r(T)$ by equation (2.9) if $\theta(T) > \bar{\theta}(\gamma)$ and $\bar{r}(\gamma)$ by if $\theta(T) \le \bar{\theta}(\gamma)$. Update the range of $\theta(T)$ as

$$\begin{cases} \theta(T)_u = \theta(T) & \text{if } \theta(T) \le \bar{\theta}(\gamma) \quad \& \quad r(T) > R \\ \theta(T)_l = \theta(T) & \text{if } \theta(T) \le \bar{\theta}(\gamma) \quad \& \quad r(T) < R \\ \theta(T)_u = \theta(T) & \text{if } \theta(T) > \bar{\theta}(\gamma) \quad \& \quad r(T) > R \\ \theta(T)_l = \theta(T) & \text{if } \theta(T) > \bar{\theta}(\gamma) \quad \& \quad r(T) < R \end{cases}$$

- Repeat the process until $r(T) = R$.

We now illustrate some Infeasible Regions for additional insights. We first show how Infeasible Regions may change with the speeds in Figures 2.3 and 2.4, where the safety distance $R = 0.5$ nmi and the chasing time $T = 2$ hours. Figures 2.3 is for the case when the speed of the pirate varies with the vessel speed v_v fixed at 20 knots, and Figure 2.4 is for the case when the vessel speed varies with the pirate speed v_p fixed at 20 knots. Both figures show that the Infeasible Region expands quickly when the pirate speed increases. This underscores the importance of maintaining a high speed of the vessel, consistent with the BMP guidance.

The range of Infeasible Regions also depends on the value of R, the safety distance. Typically it is should be no less than the effective range of the weapons that the pirate may have, for example, a few hundred meters for a rifle, and up to two thousand meters for a machine gun. The choice can be made by the vessel based on the experience and available information, such as previously reported cases of pirate attacks in the nearby area. In Figures 2.5 and 2.6, we give the Infeasible Regions under different R values for two cases, respectively, when the pirate has a higher speed and when the vessel has a higher speed.

In Figures 2.7 and 2.8, we give the Infeasible Regions under different chasing time T for two cases, respectively, when the pirate has a higher speed and when the vessel has a

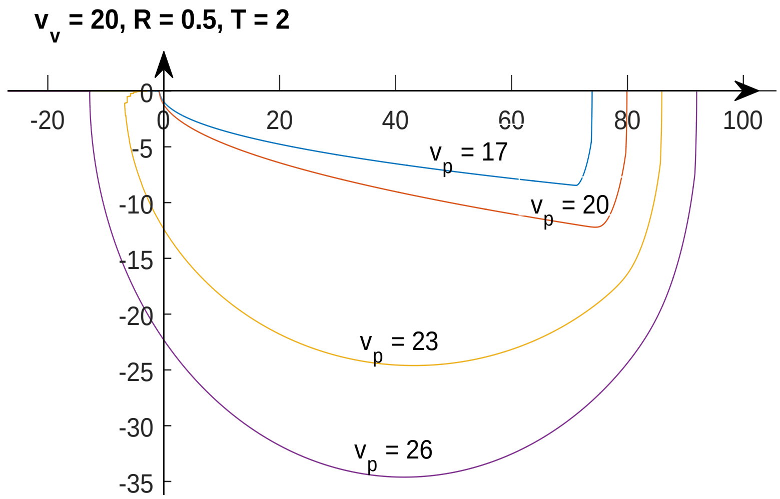


Figure 2.3: Infeasible Regions under fixed v_v

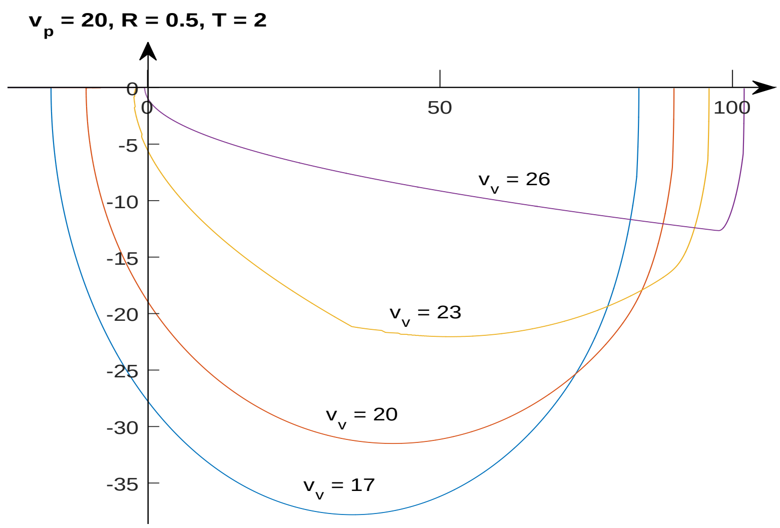


Figure 2.4: Infeasible Regions under fixed v_p

higher speed. The difference between Infeasible Regions when R varies is relatively slight. And the difference when T changes is large. Therefore, an accurate estimate of the chasing time is of huge significance to decide a feasible policy.

2.5 Policies with One or Two turns

Although direct heading is an optimal policy if it is feasible, the reality is that it is often infeasible when the pirate skiff is not found early enough. The previous numerical examples show that the Infeasible Region of the direct heading policy is quite large, especially when

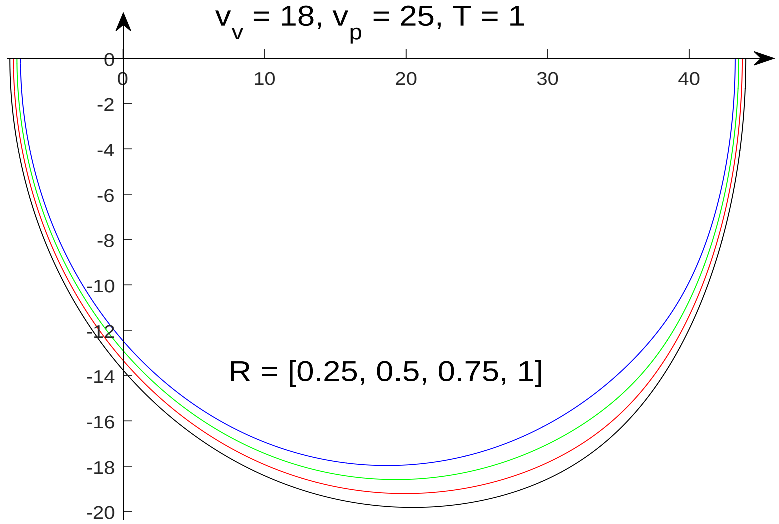


Figure 2.5: Infeasible Regions under different safety distances, when the pirate is faster

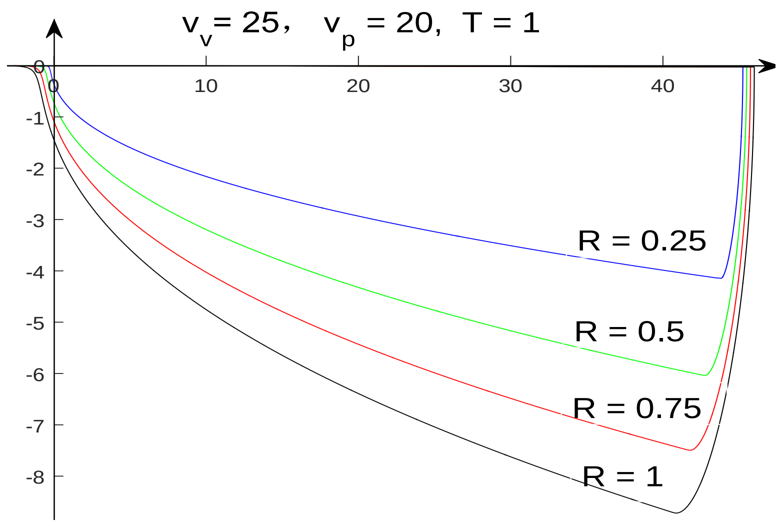


Figure 2.6: Infeasible Regions under different safety distances, when the vessel is faster

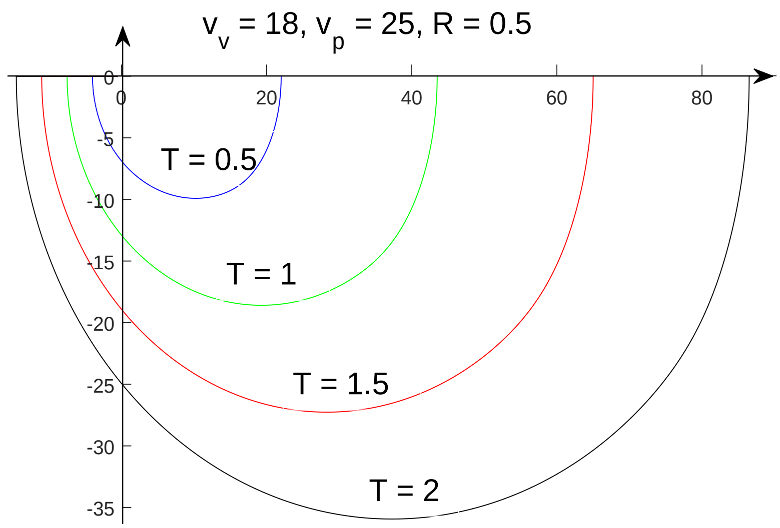


Figure 2.7: Infeasible Regions under different chasing time, when the pirate is faster

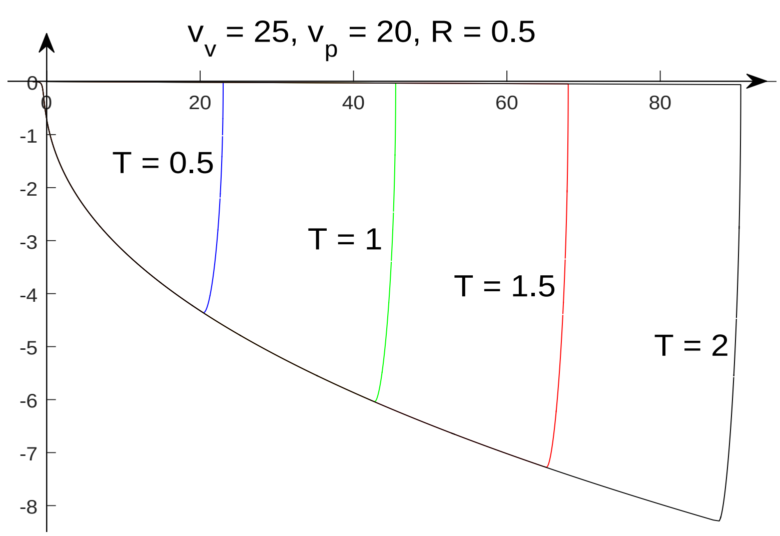


Figure 2.8: Infeasible Regions under different chasing time, when the vessel is faster

the pirate has a higher speed than the vessel. Therefore, the vessel has to change its sailing direction, i.e., to make one or even several turns.

A policy with k turns can be defined as follows. Given a maximum chasing time T, we introduce k decision points $0 = t_1 < t_2 < \cdots t_k < T$. Also denote $t_{k+1} = T$. At each decision point t_i, for $i = 1, 2, \ldots, k$, we decide a sailing direction, α_i, for the vessel during time interval $[t_i, t_{i+1}]$. The goal is to find a set of Pareto-optimal controls characterized by the turn time points $(t_1, t_2, \ldots, t_k)$ and the corresponding $(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

Theoretically speaking, the vessel can make turns at any time. However, making too many turns is not practical because the vessel may sacrifice certain time and speed to make a turn. So it is reasonable for the vessel to consider making turns as few as possible. In fact, BMP suggests a vessel "try to steer a straight course". In what follows, we will investigate two cases in detail, making one turn (Figure 2.9) and making two turns (Figure 2.10). The case of making more turns can be analyzed in a similar approach as making two turns.

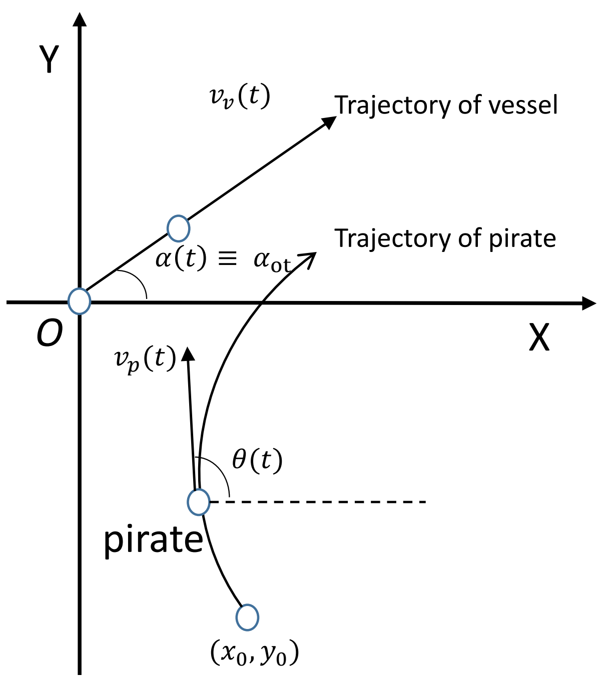


Figure 2.9: Dynamic process in one-turn policy

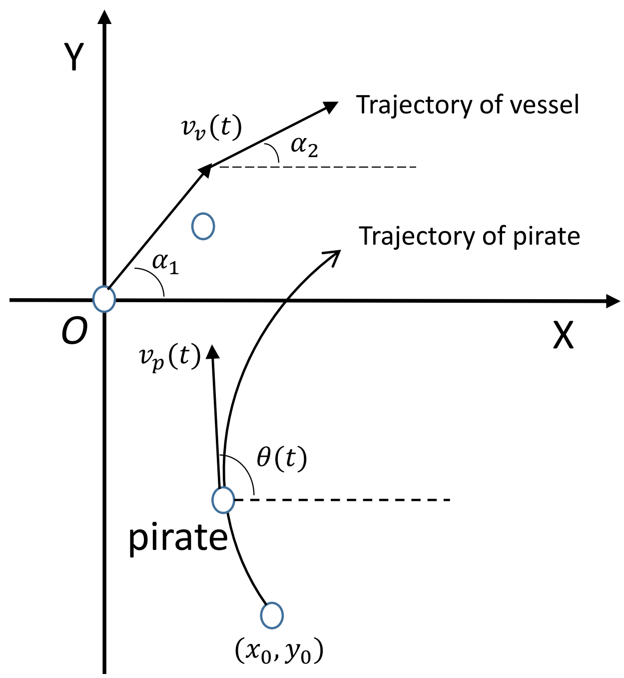


Figure 2.10: Dynamic process in two-turn policy

2.5.1 One-turn Policy

Under the one-turn policy, the vessel will turn its sailing direction to a specific α immediately after it finds being chased, and stays at this direction until T. We will refer to it as policy-α. There is another option of making only one turn where the vessel keeps the current sailing direction until time t, then turns to a new direction α and maintains the direction until time T. We will treat this option as a special case of the two-turn policy (α_1, α_2) where $\alpha_1 = 0$ and $\alpha_2 = \alpha$ and discuss more about this policy later.

Similar to the case of studying direct heading, we assume that the pirate is below or on the x-axis defined in Figure 2.1. In fact, any policy-α can be regarded as direct heading if we rotate the coordinate system anticlockwise by a degree of α. Such a view enables us to directly check the feasibility of the one-turn policy for any given α. We note that the range of α can be pretty large. However, we can only consider $\alpha \in [0, \theta_0]$. Because of the symmetry, for any $\alpha \in [0, \theta_0]$, policy-α and policy-$(2\theta_0 - \alpha)$ will result in the same $r(t)$ during $[0, T]$. Figure 2.11 shows this symmetry of the processes under policy-α and policy-$(2\theta_0 - \alpha)$.

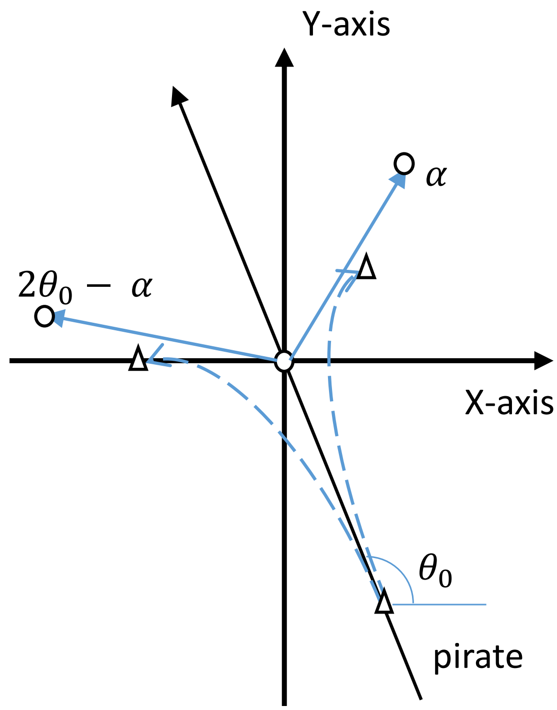


Figure 2.11: Symmetry of policy-α and policy-$(2\theta_0 - \alpha)$

Furthermore, for any $\alpha \in (0, \theta_0]$, we can also see that policy- α will dominate policy-$(2\theta_0 - \alpha)$ because, when $\theta_0 \in [0, \pi)$ as we have assumed, the policy-α always makes the vessel sailing a longer distance along the original direction. Therefore, we only need to focus on the range where $\alpha \in (0, \theta_0]$. From Lemma 2.2, the pirate will stay below the rotated x-axis, which means $\theta(t) > \alpha$, before their distance becomes zero.

Feasibility test of a policy-α can be done as follows. We use $r(t, \alpha)$ to denote the distance between the vessel and the pirate at time t, under the one-turn policy-α. From the above analysis, we know that the system status can be characterized by a modification to (2.9)-(2.11), where $\theta(t)$ is replaced by $\theta(t) - \alpha$. Hence we have

$$r(t, \alpha) = C_0(\alpha) \frac{\tan^\gamma \frac{\theta(t)-\alpha}{2}}{\sin(\theta(t) - \alpha)} \tag{2.13}$$

with $\theta(t)$ satisfying

$$-\frac{1}{2(1 + \cos(\theta(t) - \alpha))} + \frac{1}{4} \ln\Big(\frac{1 + \cos(\theta(t) - \alpha)}{1 - \cos(\theta(t) - \alpha)}\Big) = \frac{v}{C_0(\alpha)} t + C_1(\alpha), \text{if } \gamma = 1, \tag{2.14}$$

$$\frac{1}{2(\gamma - 1)} \tan^{\gamma-1} \frac{\theta(t) - \alpha}{2} + \frac{1}{2(\gamma + 1)} \tan^{\gamma+1} \frac{(\theta(t) - \alpha)}{2} = -\frac{v_v}{C_0(\alpha)} t + C_2(\alpha), \text{if } \gamma \neq 1, \tag{2.15}$$

where $C_0(\alpha) = \dfrac{r_0 \sin(\theta_0 - \alpha)}{\tan^{\gamma} \frac{(\theta_0-\alpha)}{2}}$, $C_1(\alpha) = -\dfrac{1}{2(1+\cos(\theta_0 - \alpha))} + \dfrac{1}{4}\ln(\dfrac{1+\cos(\theta_0-\alpha)}{1-\cos(\theta_0-\alpha)})$, and $C_2(\alpha) = \dfrac{1}{2(\gamma-1)}\tan^{\gamma-1}\dfrac{(\theta_0-\alpha)}{2} + \dfrac{1}{2(\gamma+1)}\tan^{\gamma+1}\dfrac{(\theta_0-\alpha)}{2}$.

From (2.13)-(2.15), we can slightly modify Algorithm 1 to check the feasibility of any one-turn policy-α. The details are omitted. We now discuss the existence of a unique optimal policy-α.

Consider any two feasible policies: policy-α and policy-α' where $0 \le \alpha < \alpha' \le \theta_0$. The final positions of the vessel under the two policies are $(x(T), y(T)) = (v_v T\cos\alpha, v_v T\sin\alpha)$ and $(x'(T), y'(T)) = (v_v T\cos\alpha', v_v T\sin\alpha')$, respectively. When $\alpha' \le \pi/2$, we always have $x(T) > x'(T) \ge 0$ and $y'(T) > y(T) \ge 0$. When $\alpha' > \pi/2$, $x'(T) < 0$. And $x(T) > x'(T)$ whenever $\alpha > \pi/2$ or $\alpha \le \pi/2$. According to Definition 2.1, α will dominate α'. It implies that there exists a unique optimal one-turn policy-α^* that dominates all other one-turn policies; specifically, α^* is the smallest among all feasible α's. The next lemma enables us to find the optimal α^*.

Lemma 2.3. *Consider two one-turn policies α and α' with $0 \le \alpha < \alpha' < \theta_0$. At any time t before $r(t,\alpha) = 0$, we have $r(t,\alpha) < r(t,\alpha')$.*

This lemma shows that if a policy-α is feasible, then any policy-α', $\alpha\prime \in [\alpha, \theta_0]$ is feasible. Hence, the optimal policy-α^* partitions the entire set $[0, \theta_0]$ into the feasible set $[\alpha^*, \theta_0]$ and infeasible set $[0, \alpha^*)$. Therefore, α^* can be found by a bisection search in $[0, \theta_0]$ as follows.

Algorithm 3.

- Initialization: $\alpha_u = \theta_0$, $\alpha_l = 0$. $\alpha = \frac{\alpha_l + \alpha_u}{2}$;

- Repeat the following steps until $\alpha - \alpha_l < \varepsilon$. Let $\theta_{new} = \theta_0 - \alpha_k$. Use Algorithm 1 to find the capture time T_c of direct heading policy given (r_0, θ_{new}). Update α_l and α_u,

$$\begin{cases} \alpha_u = \alpha, & \text{if } T_c > T; \\ \alpha_l = \alpha, & \text{if } T_c < T. \end{cases}$$

2.5.2 Two-Turn Policy

In a two-turn policy, the whole period $[0, T]$ is divided into two intervals $[0, \tau]$ and $[\tau, T]$, and the vessel has two sailing directions α_1 and α_2, as shown in Figure 2.10, such that

$$\alpha(t) = \begin{cases} \alpha_1, & \text{if } 0 \leq t \leq \tau \\ \alpha_2, & \text{if } \tau < t \leq T. \end{cases} \tag{2.16}$$

Note that τ, α_1, and α_2 are all decision variables.

Given a two-turn policy $(\tau, \alpha_1, \alpha_2)$, we can regard it as two connected one-turn policies. Similarly to the one-turn policy, we can conclude $\alpha(t) \leq \theta(t)$ for $t \in [0, T]$ due to the symmetric property of the process. Hence, we only need to consider the region where $\alpha_1 \in [\theta_0 - \pi, \theta_0]$ in the first stage and $\alpha_2 \in [\theta(\tau) - \pi, \theta(\tau)]$ in the second stage.

The final state under a two-turn policy , taking $\gamma \neq 1$ as example, satisfies the following nonlinear equations.

$$\begin{cases} r(\tau) = C_{10} \dfrac{\tan^{\gamma} \frac{\theta(\tau) - \alpha_1}{2}}{\sin(\theta(\tau) - \alpha_1)} \\ \dfrac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \dfrac{\theta(\tau) - \alpha_1}{2} + \dfrac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \dfrac{\theta(\tau) - \alpha_1}{2} = -\dfrac{v_v}{C_{10}} \tau + C_{12}, \end{cases} \tag{2.17}$$

$$\begin{cases} r(T) = C_{20} \dfrac{\tan^{\gamma} \frac{\theta(T) - \alpha_2}{2}}{\sin(\theta(T) - \alpha_2)} \\ \dfrac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \dfrac{\theta(T) - \alpha_2}{2} + \dfrac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \dfrac{\theta(T) - \alpha_2}{2} = -\dfrac{v_v}{C_{20}} (T - \tau) + C_{22}, \end{cases} \tag{2.18}$$

where

$$C_{10} = r_0 \frac{\sin(\theta_0 - \alpha_1)}{\tan^{\gamma} \frac{\theta_0 - \alpha_1}{2}}, C_{12} = \frac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \frac{\theta_0 - \alpha_1}{2} + \frac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \frac{\theta_0 - \alpha_1}{2},$$

$$C_{20} = r(\tau) \frac{\sin(\theta(\tau) - \alpha_2)}{\tan^{\gamma} \frac{\theta(\tau) - \alpha_2}{2}}, C_{22} = \frac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \frac{\theta(\tau) - \alpha_2}{2} + \frac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \frac{\theta(\tau) - \alpha_2}{2}.$$

We need to point out that one prior condition for any feasible two-turn policy to exist is the existence of feasible one-turn policies because one-turn policies include the safest and most conservative policy $\alpha = \theta_0$. However, sometimes even the optimal one-turn policy deviates from the planned route too much. Hence we hope to find two-turn policies that dominate the optimal one-turn policy.

Suppose that the optimal one-turn policy α^* has been found. Recall that α^* is the smallest feasible turning angle. Then we can eliminate some two-turn policies. First, any two-turn policy $(\tau, \alpha_1, \alpha_2)$ with $\alpha_1 < \alpha^*$ and $\alpha_2 < \alpha^*$ is infeasible regardless of the turning time τ. By using Lemma 2.3 twice, we can see that such a two-turn policy will lead to a shorter distance $r(t)$ at any time t than policy α^* does. Hence the two-turn policy is infeasible. Second, for the same reason, any two-turn policy $(\tau, \alpha_1, \alpha_2)$ with $\alpha_1 > \alpha^*$ and $\alpha_2 > \alpha^*$ is feasible but dominated by one-turn policy-α^* regardless of the turning time τ.

Based on the above analysis, we need to consider two cases: case one of $\alpha_1 > \alpha^* > \alpha_2$, and case two of $\alpha_1 < \alpha^* < \alpha_2$.

Lemma 2.4. *Assume a two-turn policy $(\tau, \alpha_1, \alpha_2)$ is taken by the vessel. If it holds that $\gamma - \cos(\theta(T; \tau, \alpha_1, \alpha_2) - \alpha_2) \geq 0$, then the final relative distance $r(T; \tau, \alpha_1, \alpha_2)$ is increasing on τ when $\alpha_1 > \alpha_2$, and decreasing on τ when $\alpha_1 < \alpha_2$.*

The condition $\gamma - \cos(\theta(T; \tau, \alpha_1, \alpha_2) - \alpha_2) \geq 0$ in Lemma 2.4 implies that the relative distance $r(t)$ is decreasing on $t \in [\tau, T]$. If $\gamma \geq 1$, $r(t)$ is strictly decreasing on $t \in [0, T]$. The sufficient condition for a two-turn policy to be feasible is $r(T; \tau, \alpha_1, \alpha_2) \geq R$. And we are able to induce the interval of feasible turn time based on Lemma 2.4. However, if $\gamma < 1$, the condition will only help to figure out whether the vessel is caught or not during $[\tau, T]$ by checking $r(T; \tau, \alpha_1, \alpha_2)$. To analyze whether the two-turn policy is feasible or not, we still need to study the minimum relative distance during the whole time interval $[0, T]$. Let $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ denote the minimum relative distance when the two-turn policy $(\tau, \alpha_1, \alpha_2)$ is taken by the vessel. Then the vessel is safe if and only if $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2) \geq R$. If $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ occurs during the time interval $(0, \tau)$, $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ is independent on τ. However, if the $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ occurs during $[\tau, T)$, $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ will depend on τ. The following lemma shows the relationship between the τ and $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ if $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ occurs during $[\tau, T)$.

Lemma 2.5. *The minimum relative distance $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ is increasing on τ when $\alpha_1 > \alpha_2$, and decreasing when $\alpha_1 < \alpha_2$.*

With these Lemma 2.4 and Lemma 2.5, we are able to conclude the feasible interval of the turn time given the turn angles α_1 and α_2.

Proposition 2.5. *For any given α_1, α_2 and the chasing time T, if a two-turn policy $(\tau, \alpha_1, \alpha_2)$ is feasible, then any control $(\tau', \alpha_1, \alpha_2)$ is feasible if $\tau' > \tau$ on the case of $\alpha_1 > \alpha^* > \alpha_2$ and $\tau' < \tau$ on the case of $\alpha_1 < \alpha^* < \alpha_2$.*

Proposition 2.5 states a threshold value for the turn time for any given α_1 and α_2. For the case of $\alpha_1 > \alpha^* > \alpha_2$, the vessel could not change its direction before a time point while the vessel could not change its direction after a time point for the case of $\alpha_1 < \alpha^* < \alpha_2$. Let $\tau^*(\alpha_1, \alpha_2)$ denote the threshold value. The feasible interval of turn time is $[\tau^*(\alpha_1, \alpha_2), T]$ in case one and $[0, \tau^*(\alpha_1, \alpha_2)]$ is case two.

After inducing the feasible interval of the turn time, we now consider the Pareto-optimal two-turn policy given α_1 and α_2. Note that the smallest relative distance will equal to R only if the turn time is $\tau^*(\alpha_1, \alpha_2)$. We can argue that for given α_1 and α_2, only the two-turn policy-$(\tau^*(\alpha_1, \alpha_2))$ is a potential Pareto-optimal two-turn policy. We take the case of $\alpha_1 > \alpha^* > \alpha_2$ as an example while similar argument can be applied to the case of $\alpha_1 < \alpha^* < \alpha_2$. When two-turn policy-$(\tau, \alpha_1, \alpha_2)$ is taken by the vessel, the final position of the vessel would be $x(T; \tau, \alpha_1, \alpha_2) = v_v \tau \cos \alpha_1 + v_v(T - \tau) \cos \alpha_2$, $y(T; \tau, \alpha_1, \alpha_2) = v_v \tau \sin \alpha_1 + v_v(T - \tau) \sin \alpha_2$. If $y(T; \tau, \alpha_1, \alpha_2) < 0$ for any α_1 and α_2, it implies that $\alpha_2 < 0$. Increasing α_2 to α_2' such that $y(T; \tau, \alpha_1, \alpha_2') = 0$, we can conclude that $x(T; \tau, \alpha_1, \alpha_2') > x(T; \tau, \alpha_1, \alpha_2)$, which means that $(\tau, \alpha_1, \alpha_2')$ dominates $(\tau, \alpha_1, \alpha_2)$. At this time, there will be no Pareto-optimal two-turn policy given α_1 and α_2. Otherwise, we have $y(T; \tau, \alpha_1, \alpha_2) > 0$. If $\alpha_2 \geq -\alpha_1$, $x(T; \tau, \alpha_1, \alpha_2)$ is monotone decreasing on τ while $y(T; \alpha_1, \alpha_2)$ is monotone increasing on τ. Thus, $(tau^*, \alpha_1, \alpha_2)$ can dominate the two-turn policies with later turn. if $\alpha_2 < -\alpha_1$, $(\tau^*, \alpha_1, \alpha_2)$ can not dominate the two-turn policies with later turn. However, for any two-turn policy with later turn, we still can find a better two-turn policy, as in Figure 2.12. In short, only the two-turn policy with earliest turn time may not be dominated by any other policies. Similarly, we can conclude that only the two-turn policy with latest turn time (Figure 2.13) may not be dominated by other two-turn policy for the case of $\alpha_1 < \alpha^* < \alpha_2$.

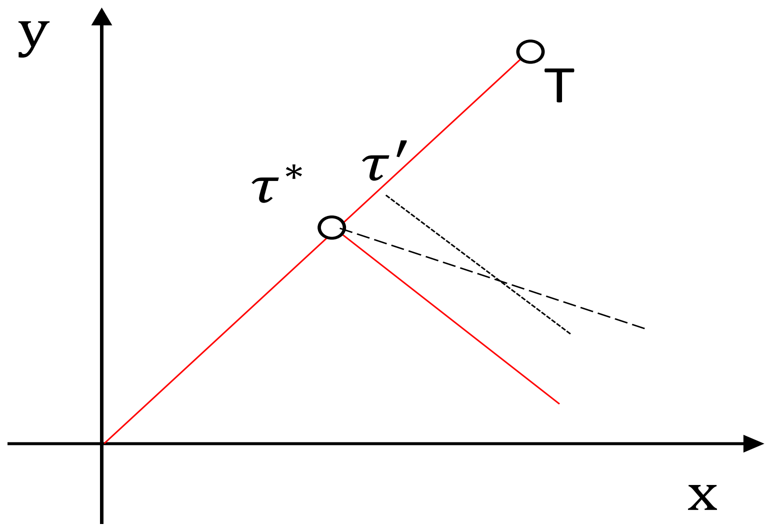


Figure 2.12: $\alpha_1 > \alpha^* > \alpha_2$, earliest turn

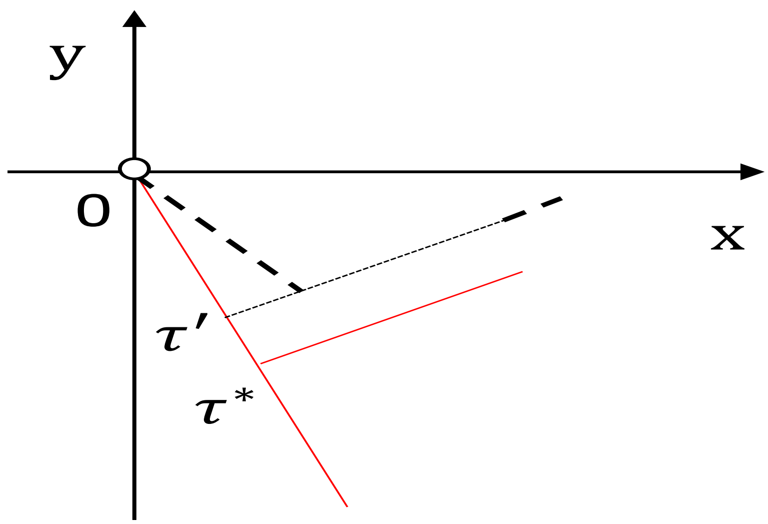


Figure 2.13: $\alpha_1 < \alpha^* < \alpha_2$, latest turn

According to Proposition 2.5, we are able to find $\tau^*(\alpha_1, \alpha_2)$ with bisection method for any fixed α_1 and α_2. The detailed procedure of finding $\tau^*(\alpha_1, \alpha_2)$ for fixed α_1 and α_2 is:

Algorithm 4.

- Initialization: $\tau_l = 0$, $\tau_u = T$; $\tau_k = (\tau_l + \tau_u)/2$;
- bisection method to find $\theta(\tau_k)$, $r(\tau_k)$ in the first stage and then $\theta(T)$, $r(T)$ in the second stage;
- Calculate $\bar{r}(\gamma; \tau, \alpha_1, \alpha_2)$ if $\gamma < 1$, based on $\theta(\tau_k)$ and $\theta(T)$;

- update t_l and t_u as following:

$$\begin{cases} \tau_u = \tau_k, \text{ if } (\tau_k, \alpha_1, \alpha_2) \text{ is feasible } \&\&\alpha_1 > \alpha_2, \\ \tau_u = \tau_k, \text{ if } (\tau_k, \alpha_1, \alpha_2) \text{ is infeasible } \&\&\alpha_1 < \alpha_2, \\ \tau_l = \tau_k, \text{ if } (\tau_k, \alpha_1, \alpha_2) \text{ is infeasible } \&\&\alpha_1 > \alpha_2, \\ \tau_l = \tau_k, \text{ if } (\tau_k, \alpha_1, \alpha_2) \text{ is feasible } \&\&\alpha_1 < \alpha_2. \end{cases}$$

2.5.3 Set of Pareto-Optimal Policies

Given α_1 and α_2, we can use Algorithm 4 to find the optimal turn time $\tau^*(\alpha_1, \alpha_2)$ and a two-turn policy $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ that dominates other two-turn policies $(\tau', \alpha_1, \alpha_2)$. By enumerating α_1 and α_2, we can identify the set of all Pareto-optimal policies computationally, making it possible to the vessel to evaluate alternative policies.

We now use some examples to demonstrate the results. First we will show the vessel's final positions $(x_v(T), y_v(T))$ under different policies, which show how two-turn policies may improve the optimal one-turn policy. In all examples, we set $v_v = 18$ knots, $v_p = 25$ knots, $R = 0.5$ nmi, and $T = 2$ hours.

Figure 2.14 shows the set of final positions $(x_v(T), y_v(T))$ of the vessel when the pirate is found at $(-10, -15)$ initially, i.e., the pirate behind the vessel. The red line is the trace of the optimal one-turn policy where $\alpha^* = 0.2324$ or 13.3^o. The yellow dots are the final positions under all Pareto-optimal two-turn policies. In this example, we see that there are indeed some two-turn policies dominating the one-turn policy-α^* with larger $x_v(T)$ and smaller $y_v(T)$, which verifies the advantage of two-turn policies. At the same time, we find that the range of different $x_v(T)$ values is quite narrow compared to the range of different $y_v(T)$ values. This shows that, in this case, the benefit of the two-turn policies is a higher chance of minimizing the deviation from the original sailing direction.

Figure 2.15 shows the set of the vessel's final positions when the pirate is found at $(10, -20)$ initially, i.e., the pirate is relatively in front of the vessel. The optimal one-turn policy is $\alpha^* = 0.7879$ or 45^o. Similar to Figure 2.14, we see the existence of two-turn policies dominating the optimal one-turn policy. There is a different observation. In Figure 2.15, the range of $x_v(T)$ values under all Pareto-optimal two-turn policies is now much bigger

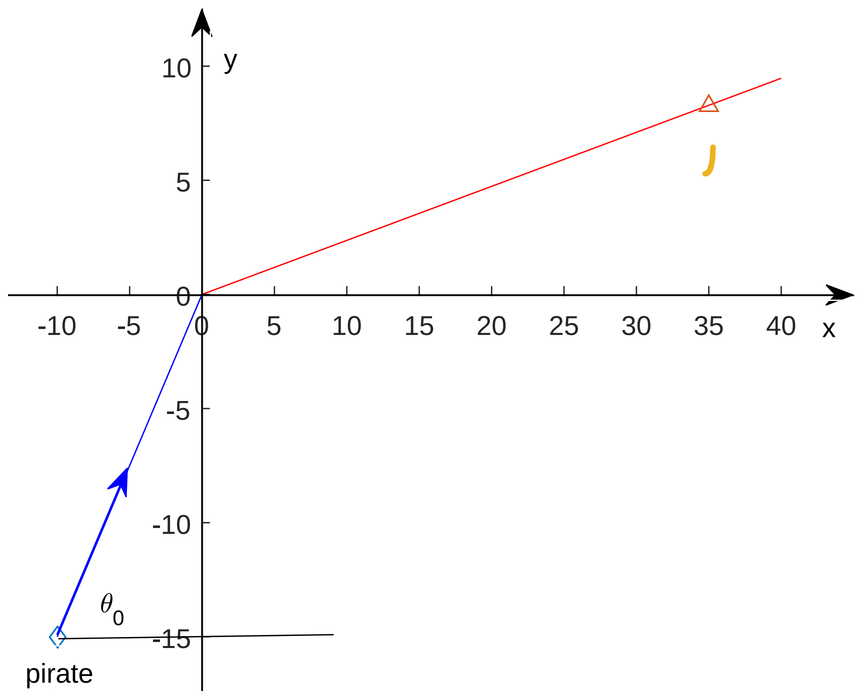


Figure 2.14: An example of two-turn policies with the pirate relatively behind the vessel

than the range of $y_v(T)$ values. In this case, the benefit of the two-turn policies is a higher chance of sailing a farther distance along the original sailing direction.

To further investigate the two-turn polices, in Figsure 2.16 and 2.17, we give the set of Pareto-optimal policies, represented by (α_1, α_2), for the two examples. Recall that we only need to consider two cases, $\alpha_1 > \alpha_2$ and $\alpha_2 > \alpha_1$. In the first case the vessel tends to evade chasing by making a larger turn in the first stage, and then tries to return to the original direction; and in the second case, the vessel tends to stick on the original direction with a smaller turn in the first stage and then tries to evade chasing. For these two examples, we can see that $\alpha_1 > \alpha_2$ for all the Pareto-optimal policies, i.e., any two-turn policy with $\alpha_1 < \alpha_2$ is dominated by some other two-turn policy with $\alpha_1 > \alpha_2$. In fact, the computational details reveal an even stronger result. We have observed that $r(T; \tau, \alpha_1, \alpha_2) \geq r(T; T - \tau, \alpha_2, \alpha_1)$ for any $\alpha_1 > \alpha_2$ and any τ. One possible explanation is that making a larger turn in the first stage will help the vessel maintain a larger the distance from the pirate.

With this observation, we now want to discuss something about the two-turn policy with $\alpha_1 = 0$ and $\alpha_2 > \alpha_1$. This special two-turn policy can be seen as one-turn policy with a turn time not being 0. With different turn time τ, we can expect that it corresponds to different

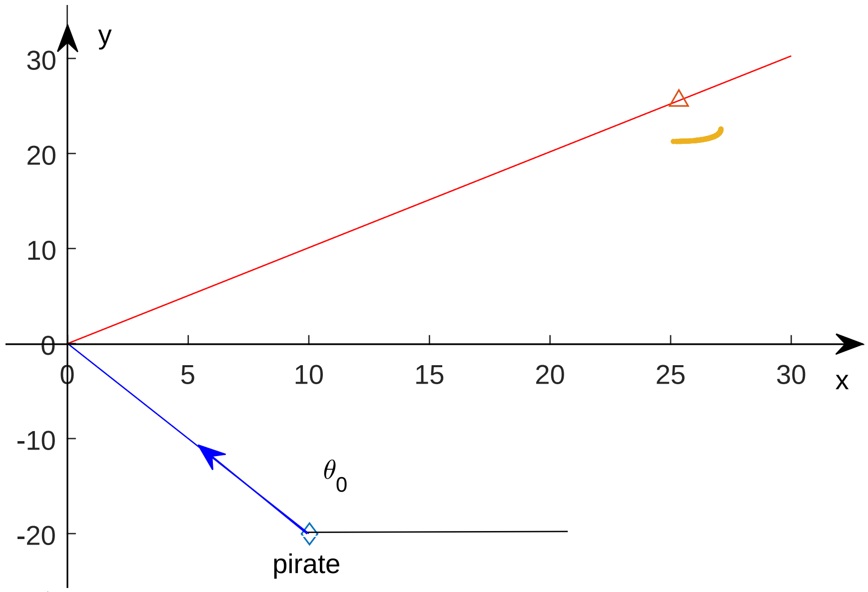


Figure 2.15: An example of two-turn policies with the pirate relatively in front of the vessel

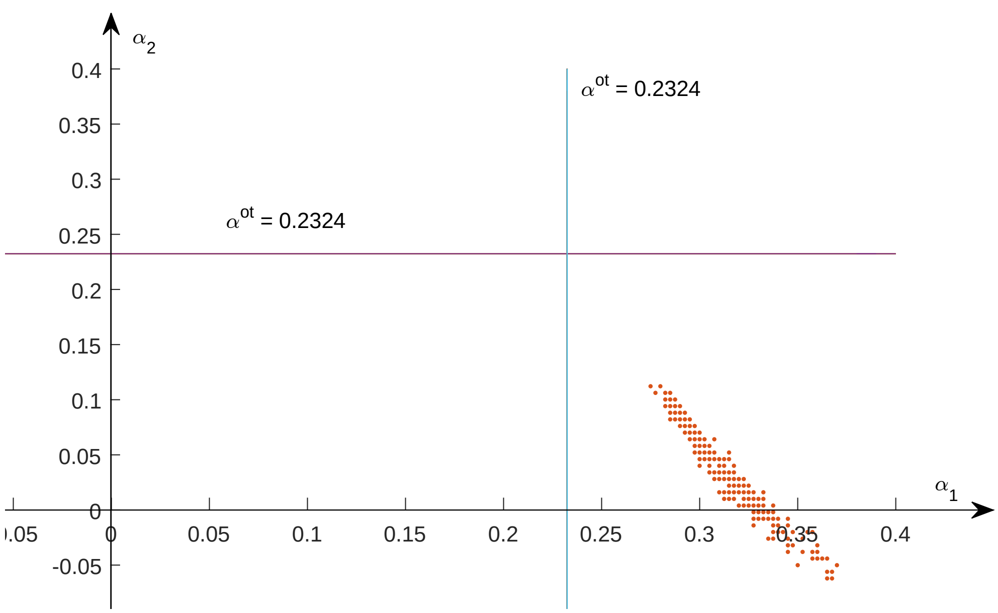


Figure 2.16: The set of Pareto-optimal policies for the example in Fig. 2.14

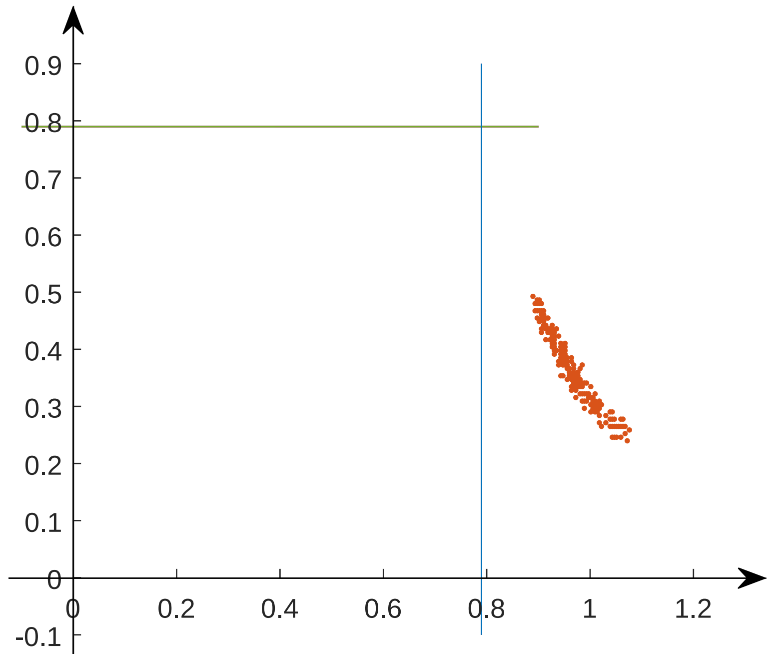


Figure 2.17: The set of Pareto-optimal policies for the example in Fig. 2.15

smallest turn angle $\alpha_2^*(\tau)$ such that the vessel could evade the chasing. However, there may not exist a two-turn policy that can dominate all other two-turn policies with $\alpha_1 = 0$ in these policies. But this observation implies that compared with a two-turn policy-$(\tau, 0, \alpha_2^*(\tau))$, $(T-\tau, \alpha_2^*(\tau), 0)$ will be safer. And thus $(\tau, 0, \alpha_2^*(\tau))$ can not be a Pareto-optimal two-turn policy.

2.6 Conclusion

Feasibility condition for the vessel to evade the chasing from pirate skiff by taking the direct heading policy is considered under different speed ratio cases. It does show that the higher speed will significantly reduce the infeasible region of the pirate skiff's position in which the vessel will be caught. In addition, the turn policy is investigated when there exist one turn and two turns. The one-turn policy is the simplest way for the vessel to take and is suggested by the BMP while the two-turn policy provides an effective way to arrive at the preferred position, such as a bigger $x(T)$ and a smaller $\mid y(T) \mid$. When evading the chasing from the pirate, the vessel should first select a large turn angle to guarantee its safety and then select a smaller turn angle to optimize its final position.

CHAPTER III

EVADING POLICIES FOR A VESSEL BEING CHASED BY MULTIPLE SKIFFS

3.1 Introduction

In Chapter 2, we have investigated three evading policies for a commercial vessel to evade the chasing from one pirate skiff. However, there used to be multiple skiffs, two skiffs in general as illustrated in [2], approaching from different direction. Assuming that each skiff takes its own pursuit guidance law, we will discuss how to check the feasibility of the one-turn and two-turn policies, and how to find the optimal policy. To simplify the presentation, we only consider the case of two chasing skiffs, but our discussion can be generalized to the case of more than two skiffs. Specifically, the generalization for the one-turn policy can be done straightforwardly, and the generalization for the two-turn policy will cause additional complexity with respect to the number of cases to consider.

Create a Cartesian coordinate system same as in Chapter 2. Let the initial positions of the skiffs be denoted by (x_1, y_1) and (x_2, y_2), or (r_1, θ_1) and (r_2, θ_2), respectively. As indicated by Figures 3.1 and 3.2, there are two cases in terms of the skiffs' positions. The two skiffs may be on the two sides of the vessel where we assume $y_1 \geq 0$ and $y_2 < 0$; they may be on the same side of the vessel where we assume $y_i \leq 0$, $i = 1, 2$. However, we can rotate the coordinate system to make the two pirate skiffs on the same side on the commercial vessel when checking the feasibility of the turn policy.

3.2 One-Turn Policy

We first discuss the one-turn policy, including direct heading as a special case. For any given turning angle α, direct heading policy-α is feasible means that it is feasible to both skiffs. This can be checked by applying Algorithm 3 twice, each to one pirate.

To find the optimal one-turn policy-α^* for the vessel, we need to known the set of all

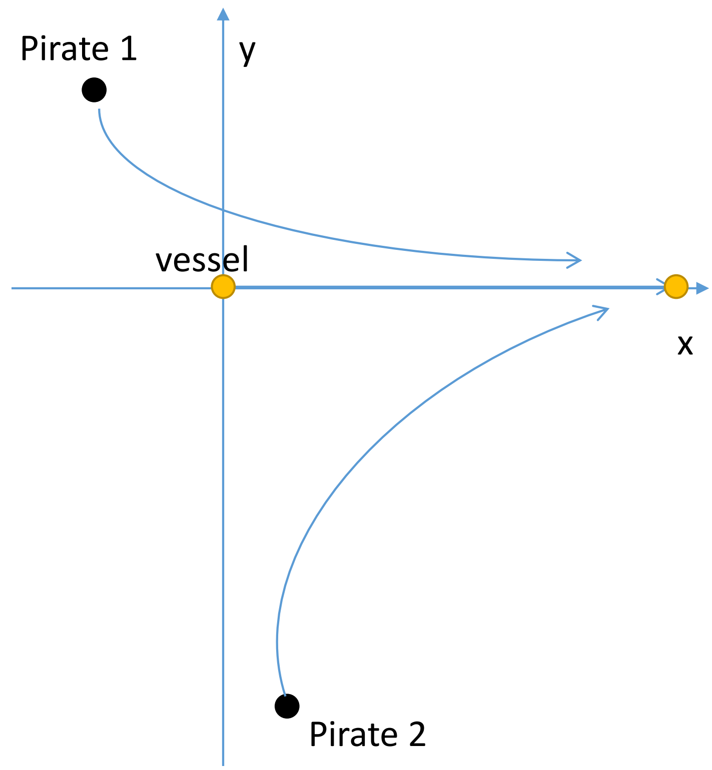


Figure 3.1: Two pirate skiffs on two sides

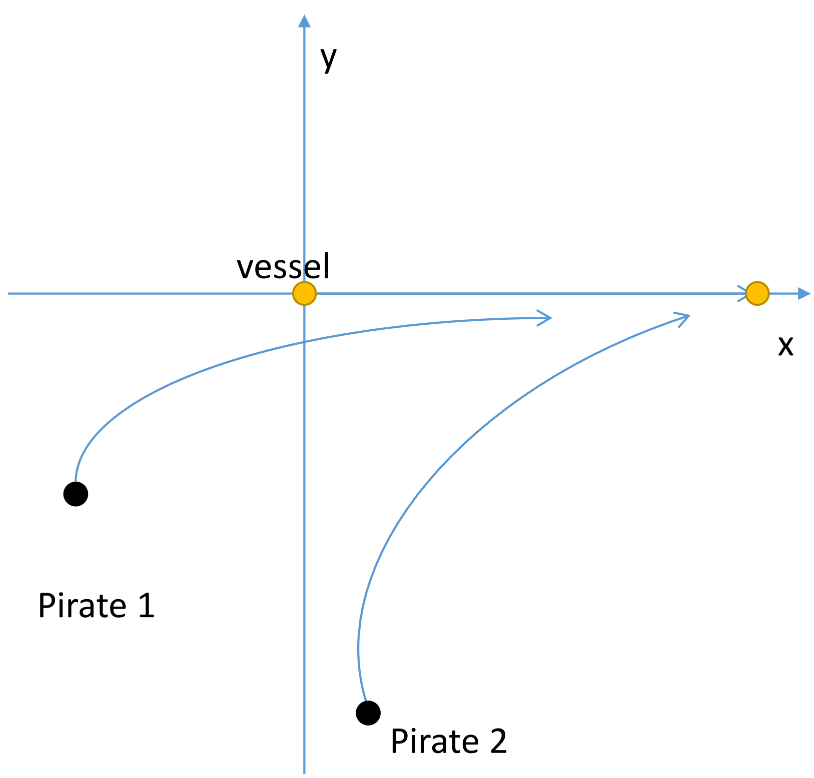


Figure 3.2: Two pirate skiffs on one side

feasible α values, denoted by A. Clearly, $A = A_1 \cap A_2$, where A_i is the set of α's feasible to skiff i. Recall that in the case of having a single chasing skiff, we point out that we only need to consider searching in $[0, \theta_0]$ to find the optimal turning angle, because any α not in $[0, \theta_0]$, even if feasible, is dominated by another counterpart feasible α' in $[0, \theta_0]$. However, now we have to consider A_i in a larger range because, though α is dominated by its counterpart α', α' may be infeasible to the other chasing skiff. So the need to find the complete set of feasible α values in $[-\pi, \pi]$, which can be done as follows. Without loss of generality, assume $y_i < 0$.

There are two cases with slight difference in determining A_i, depending on whether direct heading is feasible to skiff i.

If direct heading is infeasible to skiff i, we have Algorithm 3 to find the optimal one-turn policy α^i. According to the symmetry, $A_i = [\alpha^i, 2\theta_i - \alpha^i]$. In case $2\theta_i - \alpha^i > \pi$, we rewrite the feasible set as $A_i = [\alpha^i, \pi] \cup (-\pi, 2\theta_i - \alpha^i - 2\pi]$ such that $A_i \subset [-\pi, \pi]$.

If direct heading is feasible to skiff i, i.e., $\alpha^i = 0$. there exists feasible one-turn policy $\alpha < 0$ because $r(t, \alpha)$ is increasing on α. Let $\bar{\alpha}_i$ denote the minimum feasible α. We are able to find $\bar{\alpha}_i$ by applying Algorithm 3 with the lower bound being $\theta_i - \pi$ and upper bound being 0. The set of feasible α is now $A_i = [\bar{\alpha}_i, 2\theta_i - \bar{\alpha}_i]$ when $2\theta_i - \bar{\alpha}_i \leq \pi$ or $A_i = [\bar{\alpha}_i, \pi] \cup (-\pi, 2\theta_i - \bar{\alpha}_i - 2\pi)$ when $2\theta_i - \bar{\alpha}_i > \pi$.

In fact, there is another view of the minimum feasible α, which is related to the Infeasible Region. Let (x', y') or (r', θ') be the intersection of the cycle $\sqrt{(x^2 + y^2)} = r_i$ and the border of Infeasible Region to skiff i. According to Proposition 2.4, there will be at most one (r', θ'). If there is no intersection, any α is feasible to evade skiff i. Hence $\bar{\alpha}_i = \theta_i - \pi$. If there is an intersection, then $\theta_i > \theta'$ implies that direct heading is infeasible and $\alpha^i = \theta_i - \theta'$ while $\theta_i \leq \theta'$ means direct heading is feasible and $\bar{\alpha}_i = \theta_i - \theta'$.

Given each A_i, we have obtained A, the set of all feasible one-turn policies. Then the optimal one-turn policy-α^* can be found in A as $|\alpha^*| = \min_{\alpha \in A} |\alpha|$. This applies to any number of chasing skiffs.

There are some special cases where the optimal α^* can be identified as one α^i, the optimal one-turn policy to one of the skiffs. Consider some examples with two skiffs. When the two skiffs are on the different sides, if $\bar{\alpha}_2 \leq \alpha^1 \leq 0$, the optimal one-turn policy $\alpha^* = \alpha^1$. When the two skiffs are on the same side, if $\max\{\alpha^1, \alpha^2\} \leq \min\{2\theta_1 - \alpha^1, 2\theta_2 - \alpha^2\}$, then the optimal one-turn policy is $\alpha^* = \max\{\alpha^1, \alpha^2\}$; otherwise, there is no feasible one-turn policy.

3.3 Two-Turn Policy

When there are two skiffs, analytical properties of the two-turn policy are hard to derive because there are various cases to discuss. The complexity against the single-skiff case can be shown by the existence of feasible policies. For the problem with a single skiff, if there is no feasible one-turn policy, then there is no feasible two-turn policy either, because the one-turn policy $\alpha = \theta$ is the safest policy. For the problem with two skiffs, the non-existence of feasible one-turn policies does not necessarily imply there are no feasible two-turn policies. When there is no feasible one-turn policy with respect to two skiffs, there are actually two scenarios. First, there is no feasible one-turn policy with respect to one of the skiffs; in this case, there exists no two-turn policy with respect to the two skiffs. Second, there exist a set of feasible one-turn polices with respect to each skiff separately, but there is no overlap between the two sets; in this case, it is possible that there exist feasible two-turn policies.

The analysis of two-turn policy for the two-skiff case is still based on the two-turn policy for the one-skiff case, which includes finding an optimal turning time for a pair of turning directions and the range of feasible turning directions. We first discuss the turning time.

Given (α_1, α_2) as the two turning directions of a two-turn policy, we can find an optimal turning time $\tau^i(\alpha_1, \alpha_2)$ with respect to each single skiff i. Since usually $\tau^1(\alpha_1, \alpha_2) \neq \tau^2(\alpha_1, \alpha_2)$, we need to determine an optimal turning time $\tau^*(\alpha_1, \alpha_2)$ which is optimal with respect to the problem with two skiffs. From Proposition 2.5, we know the set of feasible turning time to skiff i, denoted by T^i, is either $[0, \tau^i(\alpha_1, \alpha_2)]$ or $[\tau^i(\alpha_1, \alpha_2), T]$, depending on the specific values of (α_1, α_2). Consequently, $\tau^*(\alpha_1, \alpha_2)$ can be determined by checking the intersection of T^1 and T^2. If the intersection is empty, then there is no feasible turning time. If the intersection is non-empty, only the two-turn policies with turn time being $\tau^i(\alpha_1, \alpha_2)$ might be Pareto-optimal two-turn policy.

We can determine the range of feasible turning directions as follows.

Consider the case where the two skiffs are on the same side of the vessel as shown in Figure 3.2. Recall that we use A_i to denote the set of feasible turning direction of the one-turn policy with respect to skiff i, where $A_i \subset (-\pi, \pi)$ and $i = 1, 2$. We assume that $\theta_2 > \theta_1$. Then Table 3.1 shows how to identify the potential sets of (α_1, α_2) of the Pareto-optimal two-turn policies. The first condition implies whether one-turn policy exists or not, in which " Yes" means one-turn policy does not exist and "No" otherwise. The other three conditions reflect the relative position of α^1 and α^2, which somehow implies which skiff is more dangerous.

Table 3.1: Possible set of (α_1, α_2) for a Pareto-optimal two-turn policy when the two skiffs are at the same side of the vessel

$\alpha^1 + \alpha^2 > 2\theta_1$	$\alpha^1 > \alpha^2$	$\alpha^2 > \theta_1$	$\theta_2 > 2\theta_1 - \alpha^1$	potential sets
Yes	/	/	/	i) $\alpha_1 \in A_1, \alpha_2 \in [\alpha^2, \theta_2]$, ii) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in A_1$
No	Yes	/	No	i) $\alpha_1 \in [\alpha^1, \theta_2], \alpha_2 \in [-\pi/2, \alpha^1]$
No	Yes	/	Yes	i) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$; ii) $\alpha_1 \in [\alpha^1, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^1]$
No	No	No	No	i) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \alpha^2]$
No	No	No	Yes	i) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$; ii) $\alpha_1 \in [\alpha^2, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^2]$
No	No	Yes	No	i) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$; ii) $\alpha_1 \in [\alpha^2, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^2]$; iii)$\alpha_1 \in [\alpha^1, \alpha^2], \alpha_2 \in [2\theta_1 - \alpha^1, \theta_2]$
No	No	Yes	Yes	i) $\alpha_1 \in [\alpha^1, \alpha^2], \alpha_2 \in [-\pi/2, \alpha^2]$

Now we will give a brief explanation of of some cases, taking the first three cases as examples. In the first case, $\alpha^1 + \alpha^2 > 2\theta_1$, i.e., $\alpha^2 > 2\theta_1 - \alpha^1$, which implies that there is no feasible one-turn policy as we assume $\theta_2 > \theta_1$. At this time, to find a feasible two-turn policy, the vessel needs to choose one direction from each A_i, $i = 1, 2$. But any feasible turn policy which contains a turn angle belonging to $[\theta_2, 2\theta_2 - \alpha^2]$ will be dominated by a smaller turn angle belonging to $[\alpha^2, \theta_2]$. Hence, the potential sets of (α_1, α_2) in the Pareto-optimal policy is i) $\alpha_1 \in A_1$, $\alpha_2 \in [\alpha^2, \theta_2]$; ii) $\alpha_1 \in [\alpha^2, \theta_2]$, $\alpha_2 \in A_1$. Meanwhile, in the second and third case, one-turn policy exists from the first condition. The second condition implies that $A_1 \subset A_2$. The optimal one-turn policy is α^1. At this time, pirate 1 is much more dangerous than pirate 2. Thus the Pareto-optimal policy mainly relies on the position of the pirate 1, and the potential set is $\alpha_1 \in [\alpha^1, \theta_2]$, $\alpha_2 \in [-\pi/2, \alpha^1]$ in the second case, and i) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$; ii) $\alpha_1 \in [\alpha^1, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^1]$ in the third case, respectively.

Secondly, the two pirate skiffs are on the two sides of the vessel. In the first case, direct heading is infeasible for the vessel to evade neither pirates. There might or not exists one-turn policy. At this time, it will be complex to reduce the potential region. Just choose one direction from each A_i, $i = 1, 2$ to find the feasible two-turn policy.

The other three cases occur when direct heading policy is assumed to be feasible to skiff 1. In the second case, the optimal one-turn policy is α^2. And it will degenerate to the problem to evade the chasing from pirate 2 only, and thus the potential set of Pareto-optimal policy is $\alpha_1 \in [\alpha^2, \theta_2]$, $\alpha_2 \in [-\pi/2, \alpha^2]$. In the third case, the optimal one-turn policy is still α^2. The whole regions can be divided into three intervals: $[-\pi/2, \alpha^2]$, $[\alpha^2, \bar{\alpha}_1]$ and $[\bar{\alpha}_1, \theta_2]$. When $\alpha_1 \in [-\pi/2, \alpha^2]$, any $\alpha_2 \in [\alpha^2, \bar{\alpha}_1]$ will be dominated by replacing the sequence of the two turn angles. Hence, we only need to consider $\alpha_2 \in [\bar{\alpha}_1, \theta_1]$. When $\alpha_1 \in [\alpha^2, \bar{\alpha}_1]$, we only need $\alpha_2 \in [-\pi/2, \alpha^2]$. When $\alpha_1 \in [\bar{\alpha}_1, \theta_2]$, the vessel needs to choose a turn angle from A_1. If $\alpha_2 \in [\alpha^2, \bar{\alpha}_1]$, such a two-turn policy will be dominated by replacing the sequence of α_1, α_2, which is definitely dominated by the optimal one-turn policy. Therefore, we only need $\alpha_2 \in [-\pi/2, \alpha^2]$. Thus we can conclude the potential sets as in the table.

Above all, we are able to conduct the computational experiments for any given two initial positions of the pirate skiffs. In fact, when we are checking the feasibility of a two-turn policy where the two pirates are on the different sides of the vessel, we may rotate the coordinate the system to make the pirates on the same side of the vessel.

The following two computational experiments help illustrate the unique feature of the two-turn policy with multiple skiffs.

Table 3.2: Potential sets of (α_1, α_2) in Pareto-optimal two-turn policy when different sides

conditions on $\alpha^1, \alpha^2, \bar{\alpha}_1$	potential sets
$\alpha^2 > 0 > \alpha^1$	i) $\alpha_1 \in A_1, \alpha_2 \in A_2$; ii) $\alpha_1 \in A_2, \alpha_2 \in A_1$
$\bar{\alpha}_1 > \theta_2$	i) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \alpha^2]$ for pirate 2 only
$\alpha^2 \leq \bar{\alpha}_1 \leq \theta_2$	i) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \alpha^2]$; ii) $\alpha_1 \in [-\pi/2, \alpha^2], \alpha_2 \in [\bar{\alpha}_1, \theta_2]$
$\bar{\alpha}_1 < \alpha^2$	i) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \bar{\alpha}_1]$; ii) $\alpha_1 \in [-\pi/2, \bar{\alpha}_1], \alpha_2 \in [\alpha^2, \theta_2]$

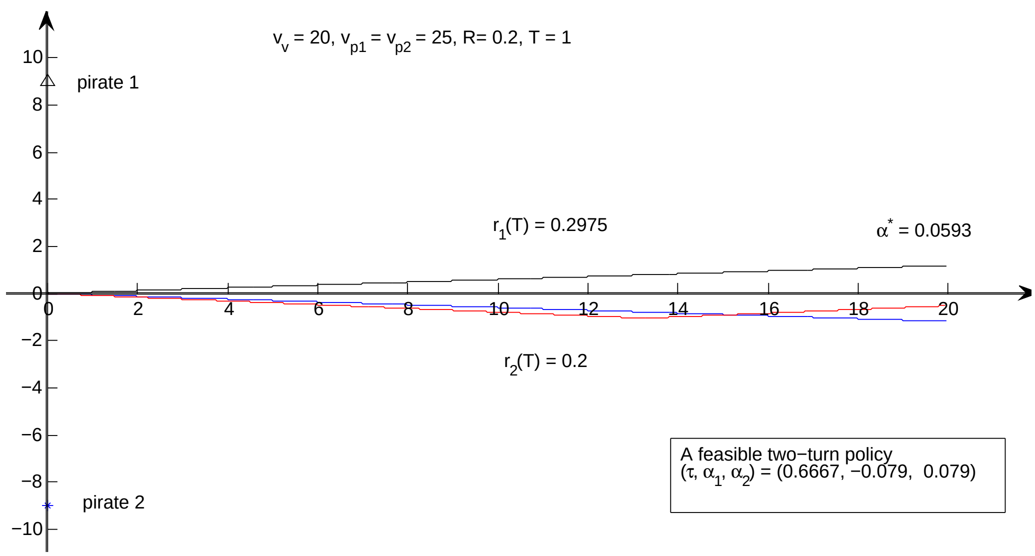


Figure 3.3: Example of feasible two-turn policy when no one-turn policy

In Figure 3.3, two spirates are found at the symmetric position $(0, 9)$ and $0, -9$. The speed of the vessel is 20 knots, while the speed of both pirate skiffs are 25 knots. We consider the safety distance as $R = 0.2$ nmi and the chasing time is $T = 1$hr. There is no one-turn policy for the commercial vessel. However, we can find a feasible two-turn policy $(0.667, -0.079, 0.079)$ for the vessel. In Figure 3.4, we fix the position of pirate 2 and change the position of pirate 1. We use the two two-turn policy which maximizes $x(T)$ or minimizes $|y(T)|$ as example. When pirate 1 is at $(-5, 20)$, direct heading policy is infeasible with respect to neighter pirate 1 nor pirate 2. There is no feasible one0turn policy and the computational result shows that there exists no feasible two-turn policy. When pirate 1 is

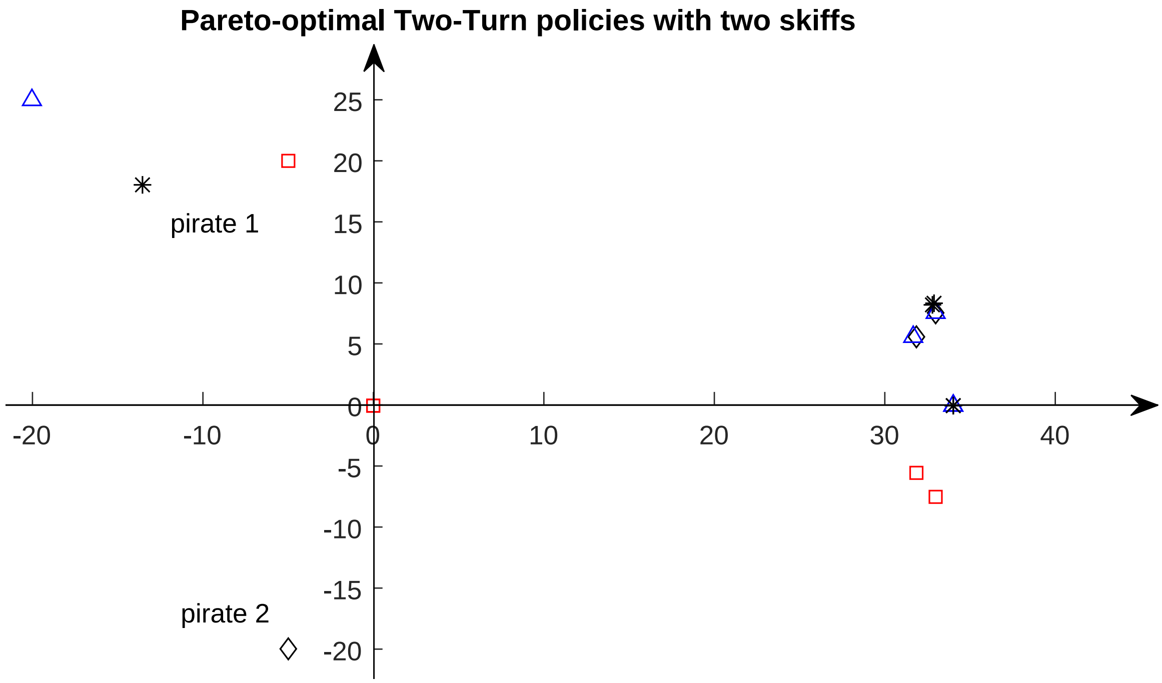


Figure 3.4: Influence of pirate's position on two-turn policies

at $(-10, 20)$, direct heading will be feasible with respect to pirate 1. However, there is still no one

3.4 Conclusion

In this paper, we extend the result in Chapter 2 to the case where there will be multiple pirate skiffs chasing the commercial vessel. Some computational experiments are conducted based on the example of two skiffs. We characterize some unique features of such a general cases from the case with only one pirate skiffs. The computational results shows that feasible two-turn policy exists even there is no one-turn policy, which illustrate the effectiveness of the turn policy.

CHAPTER IV

SPACE ALLOCATION FOR A FEEDER VESSEL WITH RESERVED AND SPOT DEMAND

4.1 Introduction

The modern liner shipping network consists of long-haul lines across oceans and feeder lines serving for a region. Typically, a long-haul line calls for a number of major ports with large container volumes, and a feeder line, with a major port as its hub port, covers smaller ports nearby to the hub port. A hub port is the transshipment point of container flows from/to the small ports on the feeder line.

A feeder line shares some common features with a long-haul line with respect to operations. For example, both follow a fixed schedule to call for ports, need to transport both laden containers and empty containers, and have demands with reservations as well as on the spot. As such, they also face similar operations planning and scheduling problems such as designing a shipping network at the tactical level, and adjusting vessel speeds at the operational level.

Due to its large volume and importance to the world economy, the liner shipping has become research direction in the transportation and logistics. There exists extensive research addressing various aspects of the liner shipping operations; e.g., see recent reviews of [8]. Nevertheless, we notice that most of such work focuses on long-haul lines, leaving feeder lines relatively understudied. This may be largely understandable because a feeder line is often of smaller scale and hence can be regarded as a simpler long-haul line.

However, a feeder line also has its unique features. For example, a feeder vessel has a hub port, a point which most containers start from or end at; there is not such a special port in long-haul lines. In addition, a feeder vessel is much smaller, usually with a capacity of hundreds, at most one or two thousand, of TEUs. Given its small size, how to effectively use the limited capacity is challenging, especially when we explicitly consider the uncertainty in the demands of containers.

Usually a vessel takes the in-advance reservation from shippers who request a certain space on the vessel. On the spot market there is also new demand without reservation. The randomness of the demand originates from both the spot market and the reserved demand. First, the demand on the spot market may come without prior information. Second, the reserved demand also suffers cancellation, which is the common practice in the container shipping industry. Shippers can reserve spaces on a vessel, but cancel some reservation without penalty; however, the feeder vessel has an obligation of fulfilling all reserved demands. In other words, the vessel has to leave enough space according to the maximum reservation even if some reservation may be canceled. This causes the capacity underutilized and opportunity loss of taking more demand from the spot market.

In this paper, we study a capacity allocation problem for a feeder vessel that faces random demands as explained above. The vessel leaves the hub port, sequentially calls for a number of ports on a feeder line, and returns the hub. At each port, the vessel collects some containers and carries them to the hub. This is a common case in the East Asia areas where the major container flow is for export. In the north America and Europe, the case is different, where a large volume of import containers need to be delivered from the hub port to regional ports. This different situation can be formulated in a similar way to our problem, but the detailed analysis will differ.

We will start with a simpler case where the vessel only collects laden containers when calling for a port. The demand at each port includes reserved demand and spot demand, both being random as explained earlier. While fulfilling the realized reservations as a commitment, the vessel needs to decide how much spot demand to take so as to balance the tradeoff between the revenue at current port and the expected revenue from the future ports. We formulate the problem as a Markov decision process, and prove that the optimal policy is to leave available capacity to the following ports no less than a threshold value. In addition, we investigate how the threshold values may change with the unit profit for serving the demand on the spot market.

We then move to a general case where the vessel not only collects laden containers but also delivers empty containers from the hub to those regional ports. Both laden and empty containers share the space of the vessel. So there is a need of coordinating the decision of laden containers collection and empty container delivery. Considering the random demand of empty containers also from reservation and spot market, we prove that the total revenue function is discretely concave, a concept extended from the continuous concavity. This enables us to establish a two-dimension-threshold policy for joint empty containers delivery and laden container collection.

From a boarder point of review, our problem belongs to the field of capacity rationing which studies how to allocate limited resource to different customers who arrive dynamically. Our problem is special in that there are two types of demands with complementary roles. While collecting laden containers consumes capacity, delivering empty containers releases capacity. This leads to a new model of capacity rationing, the contribution of our work beyond the specific collection of container shipping.

The rest of the paper is organized as follows. In section 4.2, we review the related literature. In Section 4.3, we study the problem of only collecting laden containers, and in Section 4.4 we study the problem of both laden containers collection and empty containers delivery. We conclude the paper in Section 4.5.

4.2 Literature Review

The operations of feeder lines have been largely overlooked by researchers, and only a few papers have been published in the literature. Some work has focused on feeder network design. For example, [28] investigate the design of a hub-and-spoke system including selecting a set of hub ports, allocating spoke ports to each hub port, and determining the calling sequence of the feeder vessels. [29] study a feeder network problem in which a newcomer liner service provider aims to maximize its market share against an existing liner shipping service provider. [30] study a maritime hub-and-spoke network design problem by determining the liner routing, ship size, and sailing frequency. There is also other work beyond feeder network design. For example, [31] study the problem for a terminal to serve feeder vessels, including designing preferred berthing positions and service time for cyclically visiting feeders, and allocating storage yard space between mother and feeder vessels. As far as we know, the shipping capacity utilization of a feeder vessel has not been addressed in the literature.

Our problem has one decision on delivering empty containers, which belongs to empty container repositioning, an important issue in maritime logistics, e.g.[32], [33], [34]. We refer to [33] for a comprehensive review for this line of research. The main concern of empty container repositioning has been on balancing the flow of laden and empty containers. In this paper our new constraint is the shared capacity between laden and empty containers on a vessel.

Beyond the field of liner shipping, our work is similar to vehicle routing with a predefined route, e.g., [35], [36], and [37]. Instead of deciding a visiting sequence to customers in conventional vehicle routing problem, these papers deal with a problem in which the vehicle has a given route and the decision is the timing for returning to the hub port for stock replenishment. This is different from our problem though our problem also has a fixed route.

Our problem can be regarded as a new type of capacity rationing which assigns a fixed amount capacity to dynamically arriving customers; e.g., see [38], [39], and [40]. In such problems, some available capacity is consumed whenever a customer is served. Our model is new that serving the demand of empty container delivery actually generates new available capacity for serving the demand of laden container collection.

One key technical concept in our work is the concavity of a discrete multivariate function. [41] first introduces the concept of discretely convex and concave, which will be used in our analysis. After than, several different concepts of discrete convexity have been proposed, according to different locality, such as M$-$convex function in [42], L$-$convex function in [43], M$^\natural-$convex function in [44] and L$^\natural-$convex function in [45]. All these convex functions have the property that the local optimum is the global optimum.

4.3 The Problem of Containers Collection

4.3.1 Problem Formulation

We first consider the problem where the feeder only collects laden containers from n different ports, and transports these containers back to the hub port which is referred as port 0. We list the major notation in Table 4.1.

Table 4.1: Notation

Q_i	The available capacity when the vessel arrives at port i, $i = 1, 2, \cdots, n$
$\bar{R}_i$	Reserved demand from port i before the vessel leaves hub, $i = 1, 2, \cdots, n$
R_i	Demand with reservation from port i, $i = 1, 2, \cdots, n$
r_i	A realization of R_i
S_i	Demand on spot market from port i, $i = 1, 2, \cdots, n$
s_i	A realization of S_i
p_i^r	Unit profit for serving the demand with reservation from port i, $i = 1, 2, \cdots, n$
p_i^s	Unit profit for serving the demand on spot market from port i, $i = 1, 2, \cdots, n$

Let Q_0 denote the maximum capacity for the feeder vessel. Then $Q_1 = Q_0$. Before the vessel leaves the hub port, it has accepted reservation from each port i for container collection, where the reserved number of containers is denoted as $\bar{R}_i$, $i = 1, 2, \cdots, n$. However, the realization of the demand with reservation when the vessel arrives at port i might be less than $\bar{R}_i$ due to late cancellation. We use R_i to denote the after-cancellation random demand with reservation from port i. The distributions of R_i, $i = 1, 2, \cdots, n$ are known to the feeder vessel. In addition, we use S_i to denote the demand on the spot market from port i, which is also a random variable with a known distribution.

When the vessel arrives at port i, both the demand with reservation and demand on the spot market are realized. Note that in practice the realized demand information is usually known before the actual vessel's arrival, but there is no difference in the model as long as the information is known after the vessel leaves the proceeding port. Let r_i and s_i denote the realizations of R_i and S_i, respectively. Then the vessel makes a decision on the number of containers to load at each port. Specially, the vessel has to collect all r_i containers with reservation, after that deciding a suitable number of containers to load from the spot market.

The goal of the feeder vessel is to maximize the expected revenue of the whole trip. The problem can be formulated as a Markov decision process as follows.

Let x_i^r denote the number of reserved containers to collect, and x_i^s the number of containers on the spot market to collect. Let $\pi_i(Q_i, r_i, s_i)$ denote the maximum expected revenue from port i to port n, given that the available capacity is Q_i when the vessel arrives at port i, and the realization of demand with reservation and demand on the spot market are r_i and s_i, respectively; and further let $v_i(Q_i, Q_{i+1}, r_i, s_i)$ denote the expected maximum revenue from port i to port n given that the remaining available capacity is Q_{i+1} when the vessel leaves port i. As the demand with reservation is guaranteed, we have $x_i^r = r_i$. So, $Q_{i+1} = Q_i - r_i - x_i^s$. To decide x_i^s is equivalent to decide Q_{i+1}.

Then we have a dynamic programming recursion as follows.

$$\pi_i(Q_i, r_i, s_i) = \max_{Q_{i+1}} \Big\{ v_i(Q_i, Q_{i+1}, r_i, s_i) | Q_{i+1} \geq \sum_{j=i+1}^{n} \bar{R}_j, Q_{i+1} \in \{Q_i - r_i - s_i, \cdots, Q_i - r_i\} \Big\}, \quad (4.1)$$

$$v_i(Q_i, Q_{i+1}, r_i, s_i) = (p_i^r - p_i^s) r_i + p_i^s Q_i - p_i^s Q_{i+1} + \mathbf{E}_{R_{i+1}, S_{i+1}} \pi_{i+1}(Q_{i+1}, R_{i+1}, S_{i+1}). \quad (4.2)$$

The first condition in (4.1) guarantees the full fulfillment to the demand with reservation from the future ports, and the second one reflects the range of spot demand at current port i. Consider the initial condition when the vessel arrives at port n. Since this is the last stop, the vessel will collect as many containers as possible. Hence, assuming $Q_n \geq \bar{R}_n$ to guarantee the demand with reservation from port n, we have

$$\pi_n(Q_n, r_n, s_n) = \begin{cases} p_n^r r_n + p_n^s (Q_n - r_n), & \text{if } Q_n < r_n + s_n, \\ p_n^r r_n + p_n^s s_n, & \text{if } Q_n \geq r_n + s_n. \end{cases} \quad (4.3)$$

The objective of the vessel is to maximize the expected revenue of the whole trip. Given that $Q_1 = Q_0$, the objective is thus equivalent to maximize $\pi_1(Q_0, r_1, s_1)$. Solving the above dynamic programming, we are able to figure out the serving policy for the vessel to collect containers at each port i.

4.3.2 Optimal Serving Policy

The following theorem characterizes the optimal serving policy at each port i. The policy is based on a threshold value Q_{i+1}^*. According to the remaining available capacity after serving the demand with reservation $Q_i - r_i$ and Q^*_{i+1}, the serving level to the demand on the spot market can be determined. There are three scenarios.

Theorem 4.1. *At each port i, there exists a threshold value Q^*_{i+1} for the remaining available capacity when the vessel leaves port i, such that the corresponding optimal serving policy is*

$$x_i^{s*} = \begin{cases} 0, & \text{if } Q_i - r_i \le Q^*_{i+1} \\ \min\{Q_i - r_i - Q^*_{i+1}, Q_i - r_i - \sum\limits_{j=i+1}^{n} \bar{R}_j\}, & \text{if } Q^*_{i+1} \le Q_i - r_i \le Q^*_{i+1} + s_i \\ \min\{s_i, Q_i - r_i - \sum\limits_{j=i+1}^{n} \bar{R}_j\}, & \text{if } Q_i - r_i \ge Q^*_{i+1} + s_i \end{cases}$$

In the first scenario, the remaining capacity $Q_i - r_i$ is too low compared with the threshold Q^*_{i+1}. So the vessel takes zero spot demand at port i, leaving all remaining capacity to future ports. In the second scenario, the remaining capacity is above Q^*_{i+1}, but still not high enough. The vessel can serve partial spot demand, making remaining capacity at $\min\{Q^*_{i+1}, \sum_{j=i+1}^{n} \bar{R}_j\}$ to future ports. In the third scenario, there is sufficient remaining capacity such that the vessel can take more demand from the spot market; but the remaining capacity to future ports still has a lower bound of $\sum_{j=i+1}^{n} \bar{R}_j$.

The implication of Theorem 4.1 is that the vessel needs to strategically leave certain available capacity for serving spot demand at future ports, rejecting some spot demand at current port. This is a reasonable decision when the future ports have higher unit revenue. While this seems straightforward, the theorem further points out that Q^*_{i+1}, the level of capacity left for future ports, is in independent of demand level at the current port i, which may not be intuitive.

The following theorem reveals how the threshold value Q^*_{i+1} may depend on the unit profit of the spot demand between two adjacent ports.

Theorem 4.2. *For two ports i and $i+1$, if $p_i^s < p_{i+1}^s$, then $Q^*_{i+1} > Q^*_{i+2}$; if $p_i^s \ge p_{i+1}^s$, then $Q^*_{i+1} - \bar{R}_{i+1} \le Q^*_{i+2}$.*

When $p_i^s < p_{i+1}^s$, i.e., the spot demand at port i is less profitable than that at the next port $i+1$, we should leaving certain capacity targeting for the spot demand at port $i+1$. This is indicated by $Q^*_{i+1} > Q^*_{i+2}$. When $p_i^s \ge p_{i+1}^s$, i.e., the spot demand at port i is no less profitable than that at the next port $i+1$, there is no reason to leave any capacity for the spot demand at port $i+1$. We only need to guarantee the demand with reservation at port $i+1$. This is indicated by $Q^*_{i+1} - \bar{R}_{i+1} \leq Q^*_{i+2}$.

Theorem 4.2 can be illustrated by the following example with $n = 8$ ports. At each port i, the demand with reservation R_i follows a binomial distribution $B(N, p)$ with $N = 50$, and $p = 0.9$. Note that $\bar{R}_i = N = 50$ also. Similarly, the demand on the spot market S_i follows a binomial distribution $B(N, p)$ with $N = 100$ and $p = 0.5$. The unit profit of spot demand p^s_i at each port is given in Table 4.2, and the unit profit of demand with reservation $p^r_i = 0$ as it p^r_i will not change the threshold values.

Table 4.2: Illustration of Theorem 4.2

Port No.	1	2	3	4	5	6	7	8
p^s_i	50	70	140	90	130	145	130	90
Q^*_{i+1}	663	560	282	354	213	100	50	/
$Q^*_{i+1} - \bar{R}_{i+1}$	613	510	232	304	163	50	0	/

Table 4.2 gives the corresponding threshold values. The case of $i = 1$ shows the first statement in Theorem 4.2 where $p^s_1 = 50 < p^s_2 = 70$. It shows that $Q^*_2 = 663 > Q^*_3 = 560$. The case of $i = 3$ shows the second statement where $p^s_3 = 140 > p^s_4 = 90$. We see that $Q^*_4 - \bar{R}_4 = 232 < Q^*_5 = 354$. The second statement can also be shown by the ports $i = 6, 7, 8$ where $p^s_6 > p^s_7 > p^s_8$. We can see that the vessel will only leave available space to guarantee the demand with reservation at ports 7 and 8.

4.4 Simultaneous Collection and Delivery

In practice, a vessel not only collects laden containers from each port, but also does empty container repositioning, i.e., carrying empty containers from the hub port and delivering them to regional ports. Usually transporting laden containers has a high unit profit, which is the major revenue source to the vessel. Delivering an empty container is much less profitable, even a pure cost operation to the liner. Therefore, it is reasonable to prioritize the decision of laden containers collection due to its higher importance, which justify the model studied in the above section.

Nevertheless, to make the decision more rigorous, we need an extended model that simultaneously considers the laden containers collection and empty containers delivery. These two decisions are correlated because both laden and empty containers share the common capacity of the vessel.

4.4.1 Problem Description

The feeder vessel needs to serve two types of demands, laden containers collection and empty containers delivery. To model the demand of empty containers, we propose a generic model that assumes the demand of empty containers at each port can also be classified into demand with reservation and demand from spot, similar to the case of laden containers. This model includes different scenarios as special cases. For example, there may be no empty containers reservation, or empty container reposition may be planned by the liner (and hence the demand becomes deterministic to the vessel). Our model has a high flexibility to incorporate these cases. For the same reason, we assume each empty container delivery generate some profit which may be regarded as a negative cost when empty container repositioning is a cost-only operation.

To denote the different demands, we need some new notation which is summarized in Table 4.3. We will omit the detailed explanation to each notation since the meaning is clear from the above discussion.

The four random variables R_i, S_i, D_i, and E_i will be realized when the vessel arrives at port i, where the realized values are denoted by r_i, s_i, d_i, and e_i, respectively. The feeder vessel then decides the numbers of laden containers to pick up from and empty containers to deliver to port i. Since additional space will be released after empty containers delivery, it is critical for the vessel to make decisions together.

Let x_i^r, x_i^s denote the numbers of laden containers with reservation and on spot market the vessel pick up from port i, $i = 1, 2, \cdots, n$, respectively. And let x_i^d, x_i^e denote the numbers of empty containers with reservation and on spot market it delivers to port i, $i = 1, 2, \cdots, n$, respectively. Due to the commitment that demand with reservation must be guaranteed, we have $x_i^r = r_i$ and $x_i^d = d_i$. Similar to the problem of collection only, we

Table 4.3: Notation for the extended model

Q_i^p	Available capacity when the vessel arrives at port i, $i = 1, 2, \cdots, n$
Q_i^d	Number of empty containers when the vessel arrives at port i, $i = 1, 2, \cdots, n$
$\bar{R}_i$	Reserved demand for laden containers from port i, $i = 1, 2, \cdots, n$
$\bar{S}_i$	Reserved demand for empty containers from port i, $i = 1, 2, \cdots, n$
R_i	Demand with reservation for laden containers from port i, $i = 1, 2, \cdots, n$
r_i	A realization of R_i, $i = 1, 2, \cdots, n$
S_i	Demand on spot market for laden containers from port i, $i = 1, 2, \cdots, n$
s_i	A realization of S_i, $i = 1, 2, \cdots, n$
D_i	Demand with reservation for empty containers from port i, $i = 1, 2, \cdots, n$
d_i	A realization of D_i, $i = 1, 2, \cdots, n$
E_i	Demand on spot market for empty containers from port i, $i = 1, 2, \cdots, n$
e_i	A realization of E_i, $i = 1, 2, \cdots, n$
p_i^r	Unit profit for serving laden containers with reservation from port i, $i = 1, 2, \cdots, n$
p_i^s	Unit profit for serving laden containers on spot market from port i, $i = 1, 2, \cdots, n$
p_i^d	Unit profit for serving empty containers with reservation from port i, $i = 1, 2, \cdots, n$
p_i^e	Unit profit for serving empty containers on spot market from port i, $i = 1, 2, \cdots, n$

define the maximum expected revenue $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ from port i to port n, given that the remaining available space when the vessel arrives at port i is Q_i^p, the remaining number of empty containers is Q_i^d and the realizations of demand with reservation and on spot market for laden containers and empty containers are r_i, s_i, d_i, e_i, respectively. Therefore, $Q_{i+1}^d = Q_i^d - d_i - x_i^e$ and $Q_{i+1}^p = Q_i^p + Q_i^d - Q_{i+1}^d - r_i - x_i^s$.

To decide x_i^s and x_i^e, it is equivalent to deice Q_{i+1}^p and Q_{i+1}^d. Our following discussion will be based on Q_{i+1}^p and Q_{i+1}^d to simplify the notation. Define $v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i)$ as the maximum expected revenue from port i to port n, given additional constraints that remaining available space and remaining number of empty containers are Q_{i+1}^p and Q_{i+1}^d, respectively.

Thereafter, we can obtain the dynamic programming directly as follows.

$$\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i) = \max_{Q_{i+1}^p, Q_{i+1}^d} \quad v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i) \tag{4.4}$$

$$\begin{aligned} \text{s.t.} \quad & Q_{i+1}^p \in \{Q_i^p + Q_i^d - Q_{i+1}^d - r_i - s_i, \cdots, Q_i^p + Q_i^d - Q_{i+1}^d - r_i\} \\ & Q_{i+1}^d \in \{Q_i^d - d_i - e_i, \cdots, Q_i^d - d_i\} \\ & Q_{i+1}^p \geq \sum_{j=i+1}^{n} \bar{R}_j \\ & Q_{i+1}^d \geq \sum_{j=i+1}^{n} \bar{D}_j \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} & v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i) \\ =& (p_i^r - p_i^s) r_i + (p_i^d - p_i^e) d_i + p_i^s Q_i^p + (p_i^s + p_i^e) Q_i^d - p_i^s Q_{i+1}^p - (p_i^s + p_i^e) Q_{i+1}^d \\ & + \mathbf{E}_{R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1}} \pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d, R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1}) \end{aligned} \tag{4.6}$$

The first two constraints shows the demand for laden containers and empty containers while the last two constraints is to guarantee the demand with reservation under the worst case where there will be no demand for empty containers from the future ports in the third constraint. The first six items in the $v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i)$ calculate the revenue generated from current port i and $\mathbf{E}_{R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1}} \pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d, R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1})$ represents the expected maximum revenue from the future ports. To simplify the notation, define $h_i(Q_{i+1}^p, Q_{i+1}^d) \triangleq \mathbf{E}_{R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1}} \pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d, R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1})$. Maximizing $v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i)$ is to find the trade-off between the demand on current port i and the uncertain demand from the future port.

The boundary condition for the above dynamic programming is the maximum expected revenue at the last port n, as follows ($Q_n^p \geq \bar{R}_n$ and $Q_n^d \geq \bar{D}_n$ are assumed).

$$\pi_n(Q_n^p, Q_n^d, r_n, s_n, d_n, e_n) = (p_n^r - p_n^s) r_n + (p_n^d - p_n^e) d_n + p_n^s \hat{Q}_n^p + p_n^e \hat{Q}_n^d \tag{4.7}$$

where $\hat{Q}_n^d = \min\{Q_n^d, d_n + e_n\}$ and $\hat{Q}_n^p = \min\{Q_n^p + \hat{Q}_n^d, r_n + s_n\}$

Same as in the problem of collection only, we can conclude that the serving policy is independent with p_i^r and p_i^d since the expected revenue generated by the demand with reservation from future ports is constant value independent with the serving policy for the demand on spot market. So we can simply set $p_i^r = p_i^d = 0$ for all $i = 1, 2, \cdots, n$.

Assume the initial available space and number of empty containers are Q_0^p, Q_0^d. The objective of the vessel is to maximize $\pi_1(Q_0^p, Q_0^d, r_1, s_1, d_1, e_1)$ when the vessel arrives at port 1. Furthermore, if we assume the maximum capacity of the feeder vessel is Q_0, the objective of the feeder vessel is to find the optimal (Q_0^p, Q_0^d) such that the expected revenue of the whole trip can be maximized.

4.4.2 Preliminary

Before presenting the optimal serving policy, we first introduce some technical concept used in our analysis: discretely convex function and submodular function.

There are multiple ways of defining the discrete convexity/concavity on discrete space according to different definitions of locality. In our work, we will concentrate on discretely convexity, which is first proposed by [41].

Let S be a subspace of a discrete n dimensional space $\mathbf{Z}^n$ where $\mathbf{Z}$ is the set of integers.For any real value vector $z \in \mathbf{R}^n$, its neighborhood in S is defined as $N(z) = \{u \in S : \| u - z \| < 1\}$. Here, $\| u \|$ denotes $\max_i\{u_i\}$.

Definition 4.1. *A function $f : S \to R$ is a discretely convex function if $\forall x_1, x_2 \in S$ and any $\alpha \in (0, 1)$, it holds that*

$$\alpha f(x_1) + (1-\alpha) f(x_2) \geq \min_{u \in N(\alpha x_1 + (1-\alpha) x_2)} f(u)$$

If $f(x)$ is discretely convex function, then $-f(x)$ is a discretely concave function. The following lemma in [41] introduces the locality of a discretely convex function and that local optimum is also the global optimum.

Lemma 4.1. *Consider a discretely convex function $f : S \to R$ and $x_0 \in S$. If $f(x_0) \leq f(x)$ for all $x \in S$ satisfying $\| x - x_0 \| = 1$, then $f(x_0) \leq f(x)$ for all $x \in S$, i.e., x_0 is the global minimum.*

Similarly, a local maximum of a discretely concave function is a global maximum. The following lemma gives the preservation of discretely concavity over optimization.

Lemma 4.2. *If $f(x,y) : S \to R$ is a discretely concave function, where $S = S_1 \times S_2$, $x \in S_1$, $y \in S_2$, then*

$$g(x) = \max_{y \in S_2} f(x,y)$$

is a discretely concave function.

It turns out that our profit functions are discretely concave, which is given by the following theorem.

Theorem 4.3. *For any port i, $v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i)$ and $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ are discretely concave for any (r_i, s_i, d_i, e_i).*

We also need the concept of submodularity.

Definition 4.2. *A function $f : S \to R$ is submodular if*

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \quad (\forall x, y \in S),$$

where $x \vee y = (\max\{x_i, y_i\})$ and $x \wedge y = (\min\{x_i, y_i\})$.

If a function $f : S \to R$, where $S = S_1 \times S_2$, is submodular, then for any $x^1 \in S_1$, $x^2 \in S_2$, $f(x^1 + 1, x^2) - f(x^1, x^2)$ is nonincreasing on x^2.

4.4.3 Serving Policy

At a port i, an optimal decision with respect to (Q_{i+1}^p, Q_{i+1}^d) can be made based on $(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$, which is the solution of (4.4)-(4.6). We will show that the optimal solution follows a two-dimensional threshold policy. Specifically, the decision at port i should be trying to make the vessel being at an ideal status, denoted by $(Q_{i+1}^{p*}, Q_{i+1}^{d*})$ when departing from port i; the complexity emerges if $(Q_{i+1}^{p*}, Q_{i+1}^{d*})$ is not achievable, where we should decide which one of Q_{i+1}^{p*} and Q_{i+1}^{d*} should be approached first.

Specifically, consider (4.4)-(4.6). If we relax the constraints on (Q_{i+1}^p, Q_{i+1}^d) specified in (4.5), the optimal solution will be independent of the current status $(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$. This is true because these variables have a constant contribution to (4.6) . We denote this optimal solution by $(Q_{i+1}^{p*}, Q_{i+1}^{d*})$, which is just the ideal status for the vessel to achieve when leaving port i.

The question now is how to test if $(Q^{p*}_{i+1}, Q^{d*}_{i+1})$ can be achieved. To this end, we define some notation which will help us identify different scenarios of the serving policy. Let

$$c_i \triangleq Q^p_i + Q^d_i - r_i, \quad \text{and} \quad c^*_i \triangleq Q^{p*}_{i+1} + Q^{d*}_{i+1},$$

where c_i denotes the summation of remaining available capacity and empty containers after the collection of laden containers with reservations at port i, and c^*_i is the ideal value of the summation of remaining available capacity and empty containers after the collection of laden containers from the spot market.

We first have an ideal scenario where $(Q^{p*}_{i+1}, Q^{d*}_{i+1})$ can be achieved after serving the demands from the spot market. We define $S_i \triangleq \{(Q^p_i, Q^d_i) : Q^d_i \in \{Q^{d*}_{i+1} + d_i, \cdots, Q^{d*}_{i+1} + d_i + e_i\}, c_i \in \{c^*_i, \cdots, c^*_i + s_i\}\}$. For any pair of $(Q^p_i, Q^d_i) \in S_i$, the vessel is able to achieve $(Q^{p*}_{i+1}, Q^{d*}_{i+1})$ by taking decisions as follows.

Theorem 4.4. *When $(Q^p_i, Q^d_i) \in S_i$, the optimal policy for serving the demand on the spot market is as follows.*

$$\begin{cases} x^{s*}_i = Q^p_i + Q^d_i - Q^{p*}_{i+1} - Q^{d*}_{i+1} - r_i = c_i - c^*_i \\ x^{e*}_i = Q^d_i - Q^{d*}_{i+1} - d_i. \end{cases}$$

In the general case, the vessel might not be able to achieve $(Q^{p*}_{i+1}, Q^{d*}_{i+1})$. Mathematically, when this happens, at least one of the constraint in (4.5) will be tight. With respect to the decisions of x^{s*}_i and x^{e*}_i, at least one of them will be at its boundary, either zero or the maximum possible value. To show the results, we will differentiate it into two scenarios, 1) $c_i \notin \{c^*_i, \cdots, c^*_i + s_i\}\}$, and 2) $c_i \in \{c^*_i, \cdots, c^*_i + s_i\}\}$, but $Q^d_i \notin \{Q^{d*}_{i+1} + d_i, \cdots, Q^{d*}_{i+1} + d_i + e_i\}$. We refer the former scenario as bounded collection, and the latter scenario as bounded delivery .

The following results are based on an condition that $\pi_{i+1}(Q^p_{i+1}, Q^d_{i+1}, r_{i+1}, s_{i+1}, d_{i+1}, e_{i+1})$ is a submodular function on (Q^p_{i+1}, Q^d_{i+1}). This is clearly true for $i = n$. After figuring out the optimal serving policy, we will prove that the this holds for any i with induction method.

The case of bounded collection

When $c_i \notin \{c^*_i, \cdots, c^*_i + s_i\}$, we call this as collection first scenario, which means we can find out the number of laden containers to pick up from the spot market even before delivering the empty containers. First of all, we define a problem as follows to help clarify the optimal serving policy. We slightly change the notation $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ to $\pi_i(Q_i^p, Q_i^d, \hat{c}_i, r_i, s_i, d_i, e_i)$ and $v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i)$ to $v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, \hat{c}_i, r_i, s_i, d_i, e_i)$ here.

$$
\begin{aligned}
&\pi_i(Q_i^p, Q_i^d, \hat{c}_i, r_i, s_i, d_i, e_i) = \max_{Q_{i+1}^p, Q_{i+1}^d} \quad v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, \hat{c}_i, r_i, s_i, d_i, e_i) \\
\text{s.t. } & Q_{i+1}^p + Q_{i+1}^d = \hat{c}_i; \\
& Q_{i+1}^p \geq \sum_{j=i+1}^{n} \bar{R}_j; \\
& Q_{i+1}^d \geq \sum_{j=i+1}^{n} \bar{D}_j; \\
& Q_{i+1}^p \in \{Q_i^p + Q_i^d - Q_{i+1}^d - r_i - s_i, \cdots, Q_i^p + Q_i^d - Q_{i+1}^d - r_i\} \\
& Q_{i+1}^d \in \{Q_i^d - d_i - e_i, \cdots, Q_i^d - d_i - e_i\}
\end{aligned}
\tag{4.8}
$$

Such a problem is to reallocate the total capacity $\hat{c}_i$ into a suitable number of available space for laden containers and a suitable number of empty containers to maximize the expected revenue from port i to port n. Call this problem as space reallocation problem.

Lemma 4.3. *For any given* $Q_{i+1}^p + Q_{i+1}^d = \hat{c}_i$, $v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, \hat{c}, r_i, s_i, d_i, e_i)$ *is a concave function on* Q_{i+1}^p *or* Q_{i+1}^d.

According to Lemma 4.3, we can easily found the optimal $(Q_{i+1}^p(\hat{c}_i), Q_{i+1}^d(\hat{c}_i))$ for any fixed $\hat{c}$. And we know when $\hat{c}_i = c_i^*$, $(Q_{i+1}^p(c_i^*), Q_{i+1}^d(c_i^*)) = (Q_{i+1}^{p*}, Q_{i+1}^{d*})$. In addition, we can expect that $Q_{i+1}^p(\hat{c}_i)$ and $Q_{i+1}^d(\hat{c}_i)$ are both nondecreasing on $\hat{c}_i$ if we regard the above problem as allocating the total capacity $\hat{c}_i$ into $Q_{i+1}^p(\hat{c}_i)$ and $Q_{i+1}^d(\hat{c}_i)$. If the total capacity is increasing from $\hat{c}_i$ to $\hat{c}_i + 1$, it can be regarded as one more capacity to be allocated to available space for laden containers or empty containers after allocating $\hat{c}$ into $(Q_{i+1}^p(\hat{c}_i), Q_{i+1}^d(\hat{c}_i))$.

Lemma 4.4. *Given that* $Q_{i+1}^p + Q_{i+1}^d = \hat{c}_i$, $\pi_i(Q_i^p, Q_i^d, \hat{c}_i, r_i, s_i, d_i, e_i)$ *is concave on* $\hat{c}_i$.

The optimal $\hat{c}_i$ is definitely equal to c_i^*. For any given $(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$, the feeder vessel will try to guarantee a remaining total capacity which is closest to c_i^* due to the concavity. And thus we can conclude the optimal serving policy for the collection first case.

Theorem 4.5. *If $c_i \notin \{c_i^*, \cdots, c_i^* + s_i\}$, the optimal policy for the spot demand is as following:*

$$x_i^{s*} = \begin{cases} 0 & \text{if } c_i < c_i^* \\ \min\{s_i, U_i^p\} & \text{if } c_i \geq c_i^* + s_i \end{cases}$$

$$x_i^{e*} = \begin{cases} 0 & \text{if } Q_i^d - d_i \leq Q_{i+1}^d(c_i') \\ U_i^d & \text{if } Q_i^d - d_i - e_i > Q_{i+1}^d(c_i') \\ \min\{Q_i^d - d_i - Q_{i+1}^d(c_i'), U_i^d\}, & \text{otherwise} \end{cases}$$

where $c_i' = \begin{cases} c_i \text{ if } c_i < c_i^* \\ c_i - s_i \text{ if } c_i > c_i^* + s_i \end{cases}$

and $U_i^d = \min\{e_i, Q_i^d - d_i - \sum_{j=i+1}^{n} \bar{D}_j\}$, $U_i^p = Q_i^p + d_i + U_i^d - r_i - \sum_{i+1}^{n} \bar{R}_n$.

Here, when $x_i^{s*} = L_i$, it means that the remaining number of empty containers is enough while the available space is insufficient. At this time, the feeder vessel will deliver empty containers to serve all the demand on the spot market and collect as many laden container on the spot market as possible.

The case of bounded delivery

Delivery first case occurs when $\{(Q_i^p, Q_i^d) : Q_i^d \notin \{Q_{i+1}^{d*} + d_i, \cdots, Q_{i+1}^{d*} + d_i + e_i\}, c_i \in \{c_i^*, \cdots, c_i^* + s_i\}\}$. At this time, the vessel could achieve c_i^* by picking up $c_i - c_i^*$ number of laden containers. Since $(Q_{i+1}^{p*}, Q_{i+1}^{d*})$ can not be achieved, it implies that the vessel could not deliver a suitable number of empty containers. The vessel would deliver no empty containers or all the empty containers the port requested.

If the number of empty containers to be served for the demand on the spot market is fixed as x_i^{d*}, the problem will be degenerated into the following problem ($k = Q_i^d - d_i - x_i^{e*}$ denote the remaining number of containers after serving the demand on the spot market):

$$\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i) = \max_{Q_{i+1}^p} \quad v_i(Q_i^p, Q_i^d, Q_{i+1}^p, k, r_i, s_i, d_i, e_i)$$

$$\text{s.t. } Q_{i+1}^p \geq \sum_{j=i+1}^{n} \bar{R}_j; \tag{4.9}$$

$$Q_{i+1}^p \in \{Q_i^p + Q_i^d - k - r_i - s_i, \cdots, Q_i^p + Q_i^d - k - r_i\}$$

Due to the nondecreasing first forward difference of the discretely convex function, such a problem is exactly same as the problem of collection only. Now, let $Q_{i+1}^p(k)$ denote the corresponding threshold value here.

Lemma 4.5. *If the optimal remaining available space for laden containers is $Q_{i+1}^p(k)$ when $Q_{i+1}^d = k$, then $Q_{i+1}^p(k+1) \in \{Q_{i+1}^p(k) - 1, Q_{i+1}^p(k)\}$.*

When the remaining number of empty containers increases, the optimal remaining available space for laden containers will be nonincreasing. The increased empty containers have possibility to provide additional available space, which is of course no more than the increased number of empty containers.

The following theorem illustrates the optimal serving policy for the delivery first case.

Theorem 4.6. *For given $(Q_i^p, Q_i^d) \notin S_i$ and $c_i \in \{c_i^*, \cdots, c_i^* + s_i\}$, the optimal policy for spot demand would be as following:*

$$x_i^{e*} = \begin{cases} 0 & \text{if } Q_i^d \leq Q_{i+1}^{d*} + d_i \\ \min\{e_i, U_i^{d\prime}\} & \text{if } Q_i^d \geq Q_{i+1}^{d*} + d_i + e_i \end{cases}$$

$$x_i^{s*} = \begin{cases} 0 & \text{if } Q_i^p < Q_{i+1}^p(k) + r_i - d_i - x_i^{e*} \\ \min\{s_i, U_i^{p\prime}\} & \text{if } Q_i^p \geq Q_{i+1}^p(k) + r_i + s_i - d_i - x_i^{e*} \\ \min\{Q_i^p + d_i + x_i^{e*} - r_i - Q_{i+1}^p(k), U_i^{p\prime}\} & \text{otherwise} \end{cases}$$

where $U_i^{d\prime} = Q_i^d - d_i - \sum_{j=i+1}^{n} \bar{D}_j$, $U_i^{p\prime} = Q_i^p + d_i + x_i^{e*} - r_i - \sum_{j=i+1}^{n} \bar{R}_j$ and $k = Q_i^d - d_i - x_i^{e*}$.

Theorem 4.6 actually states that it is better for the feeder vessel to get the remaining number of empty containers close to Q_{i+1}^{d*}, like that Q_i^d is smaller than the certain number

(Q_{i+1}^{d*}), the feeder vessel would not deliver empty containers for the demand on the spot market.

To achieve the optimal serving policy for both delivery first scenario and collection first scenario, we need the condition that $\pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d, r_{i+1}, s_{i+1}, d_{i+1}, e_{i+1})$. Next, we will prove the truth of this condition with induction method.

Theorem 4.7. *The maximum expected revenue function $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ is a sub-modular function on (Q_i^p, Q_i^d).*

The nonincreasing difference of $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ implies that with a higher Q_i^p or Q_i^d, the unit profit for increasing Q_i^d or Q_i^p will be nonincreasing.

4.4.4 Heuristic Policy

Though the optimal serving policy helps save time to solve the dynamic programming, it is still time consuming, especially when the number of ports increases or the total number of maximum demand increases. However, in practice, the unit profit for delivering empty containers is far less than the unit profit for transporting the laden containers back to the hub port. Therefore, the opportunity revenue by delivering empty containers will be of no significance compared with the opportunity revenue by picking up laden containers. Hence, we consider another policy where $x_i^{e*} = \min\{e_i, Q_i^d - d_i - \sum_{j=i+1}^{n} \bar{D}_j\}$. The vessel will deliver as many empty containers on the spot market as possible at each port. It is very similar to the delivery first case. With only one decision to be made, the problem can be solved very efficiently.

Now we rewrite the dynamic programming recursion as follows, to help identify the improvement of efficiency. The whole process is described into two stage.

$$\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i) = p_i^r r_i + p_i^d d_i + \pi_i^A(Q_i^p + d_i - r_i, Q_i^d - d_i, s_i, e_i) \tag{4.10}$$

while

$$\pi_i^A(Q_i^p, Q_i^d, s_i, e_i) = \max_{Q_{i+1}^p, Q_{i+1}^d} \quad p_i^s Q_i^p + (p_i^s + p_i^e) Q_i^d - p_i^s Q_{i+1}^p - (p_i^s + p_i^e) Q_{i+1}^d + h_i(Q_{i+1}^p, Q_{i+1}^d). \tag{4.11}$$

When applying the heuristic policy, the problem can be formulated as follows:

$$\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i) = p_i^r r_i + p_i^d d_i + p_i^e x_i^{e*} + \pi_i^B(Q_i^p + d_i + x_i^{e*} - r_i, Q_i^d - d_i - x_i^{e*}, s_i) \quad (4.12)$$

while

$$\pi_i^B(Q_i^p, Q_i^d, s_i) = \max_{Q_{i+1}^p} p_i^s Q_i^p - p_i^s Q_{i+1}^p + h_i(Q_{i+1}^p, Q_{i+1}^d). \quad (4.13)$$

The key challenge to figure out the optimal serving policy and heuristic policy is to implement $\mathbf{E}_{R_{i+1},S_{i+1},D_{i+1},E_{i+1}} \pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d, R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1})$. With the commitment of fulfilling the demand with reservation, the decisions could be made sequentially. When applying the optimal serving policy, the total number of calculation at each port i is $\bar{Q}_{i+1}^p \bar{Q}_{i+1}^d (\bar{R}_{i+1} + \bar{D}_{i+1} + \bar{S}_{i+1} \bar{E}_{i+1})$. On the other hand, when applying the heuristic policy, the total number of calculation at each port i is $\bar{Q}_{i+1}^p \bar{Q}_{i+1}^d (\bar{R}_{i+1} + \bar{D}_{i+1} + \bar{S}_{i+1} + \bar{E}_{i+1})$, which means the complexity can be reduced based on $\bar{E}_{i+1}$ compared with the complexity in optimal serving policy.

4.4.5 Numerical Example

The following example compares the time consuming and the maximum expected revenue while using different policy. It shows that the maximal expected revenue from the heuristic policy is almost same as the maximal expected revenue from the optimal serving policy.

We consider that there exists N identical regional ports along the predefined route. The demands with reservation R_is and D_is follow binomial distribution $B(n,p)$ where $n = \bar{R}_i = \bar{D}_i = 10$ and $p = 0.95$ for any port $i = 1, 2, \cdots, N$. And demands on spot market S_is and E_is follow binomial distribution $B(n,p)$ as well, where $n = 20$ and $p = 0.6$ for any $i = 1, 2, \cdots, N$. All p_i^rs and p_i^ds are set as 0. The unit profit p_i^es $= [4, 6, 8, 10, 12, 14, 16, 18, 20, 22]$ and p_i^ss $= [50, 70, 120, 170, 150, 170, 190, 210, 240]$. We conduct 7 experiments by increasing the number of regional ports from $N = 4$ to $N = 10$. To compare the expected revenue, we consider the initial capacities of the vessel are $Q_0^p \approx \sum_i (\bar{R}_i + \mathbf{E} S_i)$, $Q_0^d \approx \sum_i (\bar{D}_i + \mathbf{E} E_i)$. We can see that the expected revenues with given (Q_0^p, Q_0^d) under two policies are almost same while the time consuming reduces significantly.

Table 4.4: Comparison of optimal serving policy and heuristic policy

No. of Ports	Optimal serving policy		Heuristic policy	
	$\mathbf{E}\Pi$,(Q_0^p, Q_0^d)	Time	$\mathbf{E}\Pi$,(Q_0^p, Q_0^d)	Time
4	4.1061e+03,(60,60)	7.399375	4.0654e+03,(60,60)	0.466776
5	6.2368e+03,(75,75)	13.521676	6.1447e+03,(75,75)	0.945119
6	8.1485e+03,(95,95)	22.200017	8.0327e+03,(95,95)	1.502095
7	1.0256e+04,(110,110)	32.723514	1.0128e+04,(110,110)	2.206308
8	1.2739e+04,(130,130)	46.209880	1.2524e+04,(130,130)	3.208300
9	1.5357e+04,(145,145)	64.880420	1.5114e+04,(130,130)	4.653140
10	1.8333e+04,(160,160)	86.804850	1.8069e+04,(160,160)	6.115205

To apply the heuristic policy, we require the condition that $p_i^s \gg p_i^e$. To illustrate the reasonability, we conduct the following experiment. In this example, we change p_i^es into $10*[4, 6, 8, 10, 12, 14, 16, 18, 20, 22]$, which is almost same as p_i^ss. The difference between the maximum expected revenue is much bigger now than it is in Table 4.4.

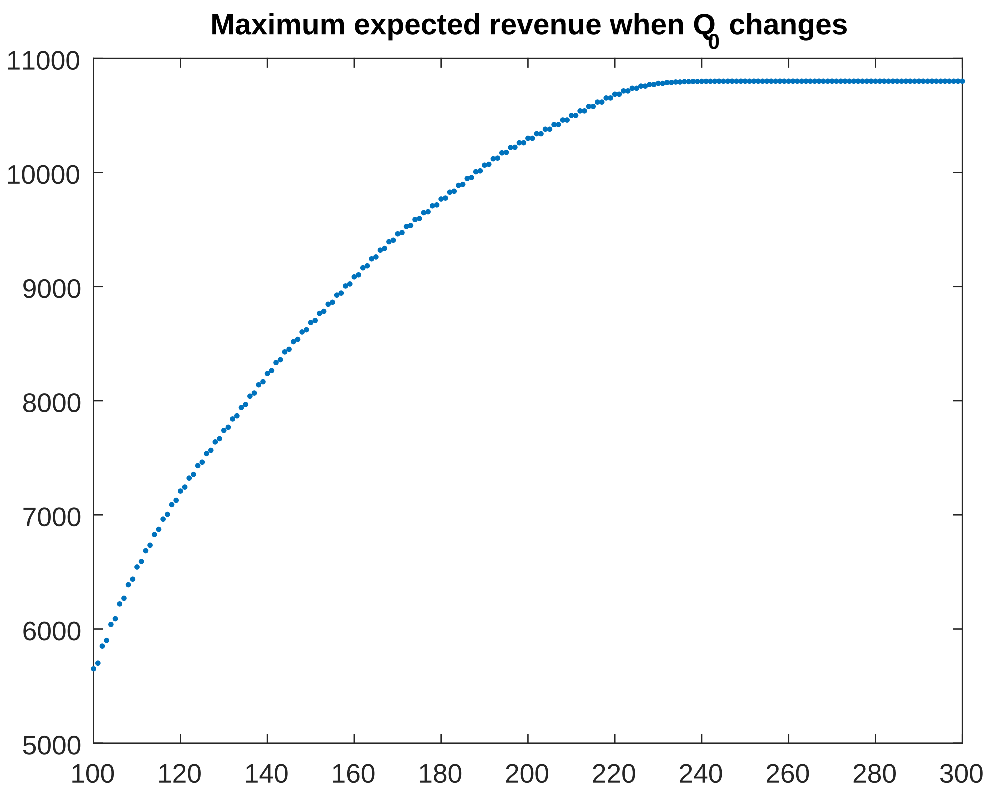
Table 4.5: Illutration of $p_i^s \gg p_i^e$ for heuristic policy

No. of Ports	5	6	7	8	9	10
optimal serving policy	8367.5	11285	13997	18310	21813	25648
heuristic policy	7447.0	10127	12717	16156	19377	23009

In addition, we also conduct an experiment to show how the maximum expected revenue depends on the initial capacity. Note that given an initial capacity of a feeder vessel, we can optimally allocate the total capacity into the empty space leaving for the laden containers to be collected from the regional ports and the space for the empty containers on board to be delivered to the regional ports, before the feeder vessel leaves the hub port. The decision is made according to the optimal serving policy for the case of bounded collection, by setting $p_0^r = p_0^s = p_0^d = p_0^e = 0$. Thus, it implies that we can calculate the maximum expected revenue during the whole trip for any given initial capacity.

In Figure 4.4.5, we present an example to show the relationship between the maximum expected revenue during the whole trip and the initial capacity. We consider there exists

$N = 5$ regional ports along the route, the first 5 ports in the above example. We can see that the maximum expected revenue is concave on the initial capacity Q_0. This is useful for the carrier to deploy a feeder vessel with suitable initial capacity, based on the estimation of the cost for deploying a feeder vessel.



4.4.6 Extensions

4.4.6.1 Skipping Port

In case the regional port doesn't make reservation for neither laden containers nor empty container, the feeder vessel only need serve the demand on the spot market. On the other hand, the feeder vessel may reject all the demand on the spot market. In other word, the feeder vessel doesn't provide any service for this regional port and thus the feeder vessel may consider to skip such a regional port. We want to discuss the condition such that the feeder vessel could skip some ports.

Assume that the feeder vessel is now at port i. Let Q_i^p and Q_i^d denote the empty space and the number of empty containers, respectively. And let r_i, s_i, d_i and e_i denote the demands. The feeder vessel now need decide whether to skip port $i+1$ or not. First of all, we need the condition that $\bar{R}_{i+1} = \bar{D}_{i+1} = 0$. Otherwise, the feeder vessel has to call for port $i+1$ to guarantee the demand with reservation. Thereafter, we have $x^{r*}_{i+1} = x^{d*}_{i+1} = 0$. When the feeder vessel decides to skip port $i+1$, it thus implies that $x^{s*}_{i+1} = x^{e*}_{i+1} = 0$ in the original optimal serving policy since the original optimal serving policy maximizes the expected revenue.

Let $(Q^{p*}_{i+1}, Q^{d*}_{i+1})$ denote the ideal state for the vessel when leaving port i and keeping calling for port $i+1$ and $(Q^{p*}_{i+2}, Q^{d*}_{i+2})$ denote the ideal state for the vessel when leaving port $i+1$. Now that we are given (Q^p_i, Q^d_i), (r_i, s_i, d_i, e_i) and $(Q^{p*}_{i+1}, Q^{d*}_{i+1})$, we can figure out the optimal serving policy for the feeder vessel to serve the port i, and assume (Q^p_{i+1}, Q^d_{i+1}) as the remaining capacity when the feeder vessel leaves port i after applying the optimal serving policy. Since $x^{s*}_{i+1} = x^{e*}_{i+1} = 0$, it thus implies one of the following two scenarios: i) The case of bounded delivery with no delivery and insufficient empty space for collecting laden containers; ii) The case of bounded collection with no collection and insufficient empty containers to deliver to port $i+1$.

In the first scenario, we define $S_1 = \{(Q^p_{i+1} + Q^d_{i+1}) : Q^d_{i+1} \leq Q^{d*}_{i+2}, Q^p_{i+1} \leq Q^p_{i+2}(Q^d_{i+1})\}$ where $Q^p_{i+2}(Q^d_{i+1})$ denotes the optimal remaining empty space for the vessel when leaving port $i+1$ given the remaining number of empty containers being Q^d_{i+1}. Here, $Q^d_{i+1} \leq Q^{d*}_{i+2}$ means no delivery for port $i+1$ while $Q^p_{i+1} \leq Q^p_{i+2}(Q^d_{i+1})$ means insufficient empty space such that the vessel would collect no laden containers. In the second scenario, We define $S_2 = \{Q^p_{i+1} + Q^d_{i+1}) : Q^p_{i+1} + Q^d_{i+1} \leq Q^{p*}_{i+2} + Q^{d*}_{i+2}, Q^d_{i+1} \leq Q^d_{i+2}(c_{i+1})\}$ where $c_{i+1} = Q^p_{i+1} + Q^d_{i+1}$ and $Q^d_{i+2}(c_{i+1})$ denote the optimal remaining empty containers given $Q^p_{i+2} + Q^d_{i+2} = c_{i+1}$ for the case of bounded collection. Here, $Q^p_{i+1} + Q^d_{i+1} \leq Q^{p*}_{i+2} + Q^{d*}_{i+2}$ implies no collection for port $i+1$ while $Q^d_{i+1} \leq Q^d_{i+2}(c_{i+1})$ implies insufficient empty containers implies. If $(Q^p_{i+1}, Q^d_{i+1}) \in S = S_1 \bigcup S_2$, then the vessel could skip port $i+1$.

Note that when the vessel decides to skip port $i+1$, the ideal state for the remaining capacity when the vessel leaves port i will change, and thus we need to revise the optimal serving policy at port i.

4.4.6.2 Rejecting Demand with Reservation

In this work, one specific condition is that the realized demand with reservation must be fulfilled. Now we want to extend into the even more general case where the vessel could also reject the demand with reservation. At this time, it will incur a penalty. For the spot demand, there could also be a loss or not for the vessel to reject the spot demand. Such a problem can be formulated as a Markov decision process as well.

Let x_i^r denote the number of rejected laden containers with reservation and x_i^d denote the number of rejected empty containers with reservation, respectively. And Moreover, let ρ_i^r and ρ_i^d denote the penalty for rejecting one unit of laden container and empty container with reservation, respectively. And let ϱ_i^s and ϱ_i^e denote the loss for unsatisfying one unit of laden container and empty container on spot market. In the above problem, we assume that $\rho_i^r = \rho_i^d = \varrho_i^s = \varrho_i^e = 0$. We have $Q_{i+1}^p = Q_i^p + Q_i^d - Q_{i+1} - (r_i - x_i^r) - x_i^s$ and $Q_{i+1}^d = Q_i^d - (d_i - x_i^d) - x_i^e$. Hence, $x_i^s = Q_i^p + Q_i^d - Q_{i+1}^d - (r_i - x_i^r) - Q_{i+1}^p$ and $x_i^e = Q_i^d - (d_i - x_i^d) - Q_{i+1}^p$.

Thereafter, the dynamic programming recursion is then as follows.

$$\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i) = \max_{x_i^r, x_i^d, Q_{i+1}^p, Q_{i+1}^d} v_i(Q_i^p, Q_i^d, x_i^r, x_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i) \quad (4.14)$$

$$\begin{aligned}
\text{s.t.} \quad & x_i^r \in \{0, \cdots, r_i\} \\
& x_i^d \in \{0, \cdots, d_i\} \\
& Q_{i+1}^p \in \{Q_i^p + Q_i^d - Q_{i+1}^d - (r_i - x_i^r) - s_i, \cdots, Q_i^p + Q_i^d - Q_{i+1}^d - (r_i - x_i^r)\} \\
& Q_{i+1}^d \in \{Q_i^d - (d_i - x_i^d) - e_i, \cdots, Q_i^d - (d_i - x_i^d)\} \qquad (4.15) \\
& Q_{i+1}^p \geq \sum_{j=i+1}^{n} \bar{R}_j \\
& Q_{i+1}^d \geq \sum_{j=i+1}^{n} \bar{D}_j
\end{aligned}$$

where

$$
\begin{aligned}
& v_i(Q_i^p, Q_i^d, x_i^r, x_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i) \\
=& (p_i^r - p_i^s - \varrho_i^s)r_i + (p_i^d - p_i^e - \varrho_i^e)d_i + (p_i^s + \varrho_i^s - \ p_i^r - \rho_i^r)x_i^r + (p_i^e + \varrho_i^e - p_i^d - \rho_i^d)x_i^d \\
& + (p_i^s + \varrho_i^s)Q_i^p + (p_i^s + \varrho_i^s + p_i^e + \varrho_i^e)Q_i^d - (p_i^s + \varrho_i^s)Q_{i+1}^p - (p_i^s + \varrho_i^s + p_i^e + \varrho_i^e)Q_{i+1}^d \\
& - p_i^s s_i - p_i^e e_i + \mathbf{E}_{R_{i+1},S_{i+1},D_{i+1},E_{i+1}} \pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d, R_{i+1}, S_{i+1}, D_{i+1}, E_{i+1})
\end{aligned}
\tag{4.16}
$$

Such a problem can be solved computationally, however, we can not point out the optimal serving policy given (Q_i^p, Q_i^d) and (r_i, s_i, d_i, e_i) since there exists four decision variables now.

4.5 Conclusion & Future Work

In this work, we consider a space allocation problem in the feeder lines, Two cases are studied: i) problem of collection only; ii) problem of collection and delivery. A predefined route is assumed for the feeder vessel during the trip. We investigate the optimal loading and unloading policy for the vessel to pick up laden containers and deliver empty containers, given random demand with known distributions. The demand is divided into two types: demand with reservation and demand on the spot market. Our future work mainly focuses on feeder network design and the frequency of feeder service.

CHAPTER V

CONCLUSIONS

To summarize, we study two issues about the maritime logistics management in this thesis. One is about the maritime security caused by piracy activities and the other is about container planning in the feeder lines.

In Chapter 2, we investigate some evading policies for a commercial vessel being chased by one pirate skiff. For the direct heading policy such that the vessel could maintain its direction, we derive the feasibility condition under different speed ratios and develop some algorithms to generate the infeasible regions. When the direct heading policy is infeasible for the vessel, we then discuss how to find a feasible one-turn and two-turn policy and how to optimize the one-turn and two-turn policy, based on the concept of Pareto optimal policy. The result shows that in the two-turn policy, the vessel should select a large turn angle at first and then select a small turn angle to move back to the planned lane.

In Chapter 3, we extend the result in Chapter 2 into the situation where there are multiple pirate skiffs chasing the commercial vessel. We investigate the optimality condition for the three policies in Chapter 2 and discuss about how to find the feasible policies. Several computational experiments are conducted and they show some interesting result.

In the future work, we may focus on the following extensions. The first one is to consider different guidance laws for the pirate skiff, like proportional navigation in Chapter 2, or the pirate could make decision upon the vessel's action. In addition, note that we assume the pirate skiffs do not cooperate when chasing the commercial vessel in Chapter 3. We may study how the cooperation between the pirate skiffs influence the commercial vessel's action.

In Chapter 4, we study a problem for a feeder vessel to collect laden containers and deliver empty containers. The route of the feeder vessel is predefined. And there are two types of demand on the market, demand with reservation which need be guaranteed and demand on spot market which induces a higher profit. We first provide the optimal serving policy if the vessel only needs to collect laden containers or deliver empty containers, and show some properties of the optimal serving policy. Thereafter, we derive the optimal serving policy where the vessel has to simultaneously collect the laden containers and empty containers.

There are several interesting research directions of such a problem. First, we may extend the problem to the case where the feeder vessel would serve the feeder line periodically, according to a fixed schedule. Under the dynamic process, the demand not satisfied during this trip could be backlogged. Then it comes a question how to determine the optimal serving frequency, and may even the relationship between the container capacity of the feeder vessel and the serving frequency. Second, we may consider the design of a feeder network. Due to the limited capacity of a feeder vessel, the feeder vessel may have to decline the most of the demand on the spot market. Then it may be possible to divide these ports into several clusters serving by multiple feeder vessels, and thus most of the demand could be satisfied. The way to divide the ports and the capacity of the feeder vessel is worth deep investigation.

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APPENDIX A

TECHNICAL SUPPORTS FOR CHAPTER II

A.1 Proof of Lemma 2.1

From (2.2), the dynamic process of $r(t)$ can be described by the following differential equation

$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} = \frac{2(x_v(t) - x_p(t))(\frac{\mathrm{d}x_v(t)}{\mathrm{d}t} - \frac{\mathrm{d}x_p(t)}{\mathrm{d}t}) + 2(y_v(t) - y_p(t))(\frac{\mathrm{d}y_v(t)}{\mathrm{d}t} - \frac{\mathrm{d}y_p(t)}{\mathrm{d}t})}{2r(t)}.$$

From (2.3), we can simplify it to

$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} = v_v(t)\cos(\alpha(t) - \theta(t)) - v_p(t)\cos(\beta(t) - \theta(t)). \tag{A.1}$$

Therefore, with respect to $\beta(t)$, $\frac{\mathrm{d}r(t)}{\mathrm{d}t}$ is minimized when $\beta(t) = \theta(t)$.

A.2 Proof of Lemma 2.2

We first consider the case $\gamma > 1$.

Consider (2.11). Since $\lim_{t\to(\frac{C_0C_2}{v_v})^-}$ rhs of (2.11) $= 0$, we have $\lim_{t\to(\frac{C_0C_2}{v_v})^-}$ lhs of (2.11) $= 0$. This implies that $\lim_{t\to(\frac{C_0C_2}{v_v})^-}\theta(t) = 0$ because the two terms of the lhs of (2.11) are both non-negative. Moreover, $\lim_{t\to(\frac{C_0C_2}{v_v})^-} r(t) = \lim_{t\to(\frac{C_0C_2}{v_v})^-} C_0\frac{\tan^\gamma\frac{\theta(t)}{2}}{\sin\theta(t)} = \lim_{t\to(\frac{C_0C_2}{v_v})^-} C_0\frac{\sin^{\gamma-1}\theta(t)}{(1+\cos\theta(t))^\gamma} = 0$. Besides, both $r(t)$ and $\theta(t)$ are strictly decreasing on t when $\theta(t) > 0$, and $\theta(t) = 0$ after $\theta(t) = 0$ since $\frac{d\theta(t)}{dt} = 0$ before $r(t) = 0$. Therefore, $\theta(t) > 0$ when $t < \frac{C_0C_2}{v_v}$ and $\theta(t) = 0$ when $t \geq \frac{C_0C_2}{v_v}$. Hence, we can conclude that $\tau = \frac{C_0C_2}{v_v}$ by the definition of τ.

Next, we will prove the non-existence of τ when $\gamma < 1$ by a contradiction.

We first assume such a finite τ exists for any given $\gamma < 1$, and the corresponding relative distance is denoted by $r(\tau)$. As τ is finite, the reduced relative distance will be no more than $v_v\tau(\gamma - \cos\theta_0)$, which implies $r(\tau)$ must be finite as well.

When $\gamma = 1$, we prove a contradiction by considering the continuity of $\theta(t)$. Note that $\lim_{t\to\tau^-}\theta(t) = 0$ implies $\lim_{t\to\tau^-}$ lhs of (2.10) $= +\infty$. In addition, $\lim_{t\to\tau^-}$ rhs of (2.10) $= \lim_{t\to\tau^-}\frac{v_vt}{C_0} +$

$C_1 = \dfrac{v_v\tau}{C_0} + C_1$. Since τ is finite, $\frac{v_v\tau}{C_0} + C_1$ must be finite. Therefore, $\lim_{t\to\tau^-}$ rhs of (2.10) $\neq$ $\lim_{t\to\tau^-}$ lhs of (2.10), which implies a contradiction and thus such a finite τ doesn't exist.

When $\gamma < 1$, we prove a contradiction by considering the continuity of $r(t)$. As $\lim_{t\to\tau^-}\theta(t) = 0$, $r(t) = C_0\dfrac{\tan^\gamma\frac{\theta(t)}{2}}{\sin\theta(t)}$ for $t \in [0, \tau^-]$. Due to the continuity of $r(t)$, we thus have $\lim_{t\to\tau^-} r(t) = \lim_{t\to\tau^-} C_0\dfrac{\tan^\gamma\frac{\theta(t)}{2}}{\sin\theta(t)} = \lim_{t\to\tau^-} C_0\dfrac{\sin^{\gamma-1}\theta(t)}{(1+\cos\theta(t))^\gamma} = \lim_{t\to\tau^-}\dfrac{C_0}{\sin^{1-\gamma}\theta(t)(1+\cos\theta(t))^\gamma} = +\infty$, while we have concluded that $r(\tau)$ is finite. Therefore, $\lim_{t\to\tau^-} r(t) \neq r(\tau)$, which implies a contradiction and thus finite τ does not exist.

A.3 Proof of Proposition 2.1

When $\gamma = 1$, (2.9) becomes

$$r(t) = C_0\frac{\tan\frac{\theta(t)}{2}}{\sin\theta(t)} = C_0\frac{\frac{\sin\theta(t)}{1+\cos\theta(t)}}{\sin\theta(t)} = \frac{C_0}{1+\cos\theta(t)}. \tag{A.2}$$

Therefore, when $C_0 = r_0(1+\cos\theta_0) \geq 2R$, we have $r(t) > \frac{r_0(1+\cos\theta_0)}{2} \geq R$ for $t \in [0, +\infty)$, i.e., T_c does not exist.

When $C_0 < 2R$, we can verify that

$$t = T_c = \frac{r_0 - R}{2v_v} + \frac{r_0(1+\cos\theta_0)}{4v_v}\ln\frac{r_0(1-\cos\theta_0)}{2R - r_0(1+\cos\theta_0)} \tag{A.3}$$

is a solution of (A.2) and (2.10). It is also the unique solution because (2.7) implies $r(t)$ is strictly decreasing, i.e., $r(t) > 0$ for $t \in [0, T_c]$, and $r(T_c) = R$.

A.4 Proof of Proposition 2.2

When $\gamma > 1$, $\dfrac{\mathrm{d}r(t)}{\mathrm{d}t} = v_v(\cos\theta - \gamma) < 0$ for any $t \geq 0$, which means the distance $r(t)$ between the vessel and the pirate will keep decreasing until $r(t) = 0$. From (2.11), $\lim_{t\to\frac{r_0(\gamma+\cos\theta_0)}{\gamma^2-1}} r(t) = 0$. So we can claim that $T_c \in \big[0, \frac{r_0(\gamma+\cos\theta_0)}{\gamma^2-1}\big)$.

A.5 Proof of Proposition 2.3

1) Since $r(t)$ is a convex function on t, we can solve the first-order condition $\dfrac{\mathrm{d}r(t)}{\mathrm{d}t} = v_v(\cos\theta(t) - \gamma) = 0$, which shows $r(t)$ is minimized at $\cos\theta(t) = \gamma$, or equivalently, $\theta(t) = \bar{\theta}(\gamma)$. Substitute $\theta(t)$ by $\bar{\theta}(\gamma)$ in (2.11), we can get

$\bar{t}(\gamma) = -\dfrac{C_0}{v_v}\Big[\dfrac{1}{2(\gamma-1)}\tan^{\gamma-1}\dfrac{\bar{\theta}(\gamma)}{2} + \dfrac{1}{2(\gamma+1)}\tan^{\gamma+1}\dfrac{\bar{\theta}(\gamma)}{2} - C_2\Big] = \dfrac{r_0(\gamma+\cos\theta_0) - 2\gamma\bar{r}(\gamma)}{v_v(\gamma^2-1)}$, which is the only solution to $\cos\theta(t) = \gamma$ due to the strict monotonicity of $\theta(t)$ on t. Therefore, $r(t)$ achieves the global minimum at $\bar{t}(\gamma)$, and from (2.9), we can get the minimum distance $r(\bar{t}(\gamma)) = \bar{r}(\gamma)$.

2) If $\theta_0 \leq \bar{\theta}(\gamma)$, we have $\theta(t) \leq \theta_0$ and $\cos\theta(t) \geq \cos\theta_0 \geq \gamma$ for $\forall t \in [0, +\infty)$ because $\theta(t)$ is a decreasing function on t. Therefore, $r(t)$ will be increasing on $t \in [0, +\infty)$. Due to the assumption $r_0 \geq R$, T_c does not exist.

If $\theta_0 > \bar{\theta}(\gamma)$ but $\bar{r}(\gamma) \geq R$, from 1) we know T_c does not exist because $\bar{r}(\gamma)$ is the minimum of $r(t)$.

3) Also from 1), if $\theta_0 \leq \bar{\theta}(\gamma)$ and $\bar{r}(\gamma) < R$, we have $T_c < \bar{t}(\gamma)$.

A.6 Proof of Proposition 2.4

To simplify the notation, we replace $r(T, r_0, \theta_0)$ and $\theta(T, r_0, \theta_0)$ by $r(T)$ and $\theta(T)$ in the following proofs.

When $\gamma \geq 1$, the sufficient and necessary condition for the vessel to be safe is degenerate to $r(T) \geq R$. Thus, to prove the proposition is to prove that $r(T)$ is increasing on r_0 with fixed θ_0 and decreasing on θ_0 with fixed r_0.

We provide the details of proof when $\gamma > 1$. Same analysis can be done with $\gamma = 1$ and the conclusion will be the same.

We first prove the property for $r(T, r_0, \theta_0)$. To simplify the notation, let $r(T)$ and $\theta(T)$ denote the $r(T, r_0, \theta_0)$, respectively.

The implicit function of $\theta(T)$ is now

$$\begin{aligned}
&\frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta(T)}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta(T)}{2} \\
= &-\frac{v_v T \tan^{\gamma}\frac{\theta_0}{2}}{r_0 \sin\theta_0} + \frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta_0}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta_0}{2}
\end{aligned}$$

Taking the derivative with respect to θ_0 on both sides, we have

$$\frac{\tan^{\gamma}\frac{\theta(T)}{2}}{\sin^2\theta(T)}\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0} = \frac{\tan^{\gamma}\frac{\theta_0}{2}}{\sin^2\theta_0} - \frac{v_v T(\gamma - \cos\theta_0)}{r_0}\frac{\tan^{\gamma}\frac{\theta_0}{2}}{\sin^2\theta_0}. \tag{A.4}$$

On the other hand, we have

$$r(T) = \frac{r_0 \sin\theta_0}{\tan^{\gamma}\frac{\theta_0}{2}}\frac{\tan^{\gamma}\frac{\theta(T)}{2}}{\sin\theta(T)}.$$

Similarly, take the derivative with respect to θ_0 on both sides, we have

$$\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0} = r_0\frac{\cos\theta_0-\gamma}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta(T)}{2}}{\sin\theta(T)} + r_0\frac{\sin\theta_0}{\tan^\gamma\frac{\theta_0}{2}}\frac{(\gamma-\cos\theta(T))\tan^\gamma\frac{\theta(T)}{2}}{\sin^2\theta(T)}\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0}$$

Substituting $\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0}$ into $\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0}$, we can rewrite $\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0}$ as

$$\begin{aligned}\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0} &= \frac{r(T)(\cos\theta_0-\gamma)}{\sin\theta_0} + \frac{r_0(\gamma-\cos\theta(T))}{\sin\theta_0}[1-\frac{v_vT}{r_0}(\gamma-\cos\theta_0)] \\ &= \frac{r_0(\gamma-\cos\theta(T)) - r(T)(\gamma-\cos\theta_0) - v_vT(\gamma-\cos\theta_0)(\gamma-\cos\theta(T))}{\sin\theta_0}.\end{aligned}$$

Similarly, we can obtain the derivative of $r(T)$ with respect to r_0, which is

$$\frac{\mathrm{d}r(T)}{\mathrm{d}r_0} = \frac{r(T)+v_vT(\gamma-\cos\theta(T))}{r_0}.$$

When $\gamma > 1$, we can rewrite the $\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0}$ as following:

$$\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0} = \frac{(r_0-r(T)-v_vT(\gamma-\cos\theta_0))(\gamma-\cos\theta(T)) + r(T)(\cos\theta_0-\cos\theta(T))}{\sin\theta_0}$$

Since $r(t)$ is a monotone decreasing function on $t \in [0, T]$, we can expect $r_0 \le r(T)+v_vT(\gamma-\cos\theta_0)$. Besides, $\theta(t)$ is a monotone decreasing function as well. Therefore, $\theta_0 > \theta(T)$ and $\cos\theta_0 < \cos\theta(T)$. Hence, $\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0} < 0$.

And $\frac{\mathrm{d}r(T)}{\mathrm{d}r_0} > 0$, directly from $\gamma > 1$.

On the other hand, when $\gamma < 1$, the sufficient condition will includes two scenarios: if $\bar{t}(\gamma) < T$, $\bar{r}(\gamma) \ge R$; if $\bar{t}(\gamma) \ge T$, $r(T) \ge R$.

Note that $\bar{t}(\gamma) < T$ implies $\cos\theta(T) - \gamma > 0$.

Since $\bar{r}(\gamma, r_0, \theta_0) = r_0\frac{\sin\theta_0}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta_\gamma}{2}}{\sin\theta_\gamma}$. the derivatives of $\bar{r}(\gamma, r_0, \theta_0)$ with respect to r_0 and θ_0 are as following:

$$\frac{d\bar{r}(\gamma, r_0, \theta_0)}{dr_0} = \frac{\sin\theta_0}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta_\gamma}{2}}{\sin\theta_\gamma},$$

$$\frac{d\bar{r}(\gamma, r_0, \theta_0)}{d\theta_0} = \frac{r_0(\cos\theta_0-\gamma)}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta_\gamma}{2}}{\sin\theta_\gamma}.$$

We have $\frac{d\bar{r}(\gamma, r_0, \theta_0)}{dr_0} > 0$ and $\frac{d\bar{r}(\gamma, r_0, \theta_0)}{d\theta_0} < 0$. Note that the case $\cos\theta_0 - \gamma \ge 0$ is excluded since $\bar{r}(\gamma, r_0, \theta_0)$ will have no meaning at that time.

When $\bar{t}(\gamma) \ge T$, it then means $\cos\theta(T) - \gamma < 0$.

We can directly conclude that $\frac{\mathrm{d}r(T)}{\mathrm{d}r_0} = \frac{r(T) + v_v T(\gamma - \cos\theta(T))}{r_0} > 0$.

Next, we will prove $\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0} < 0$.

At this time, $r(t)$ will be first decreasing and then increasing. Considering $\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0}$ at first. Note that $\theta(T)$ is increasing when $r_0 - v_v T(\cos\theta_0 - \gamma) > 0$, i.e., $\theta_0 < \arccos(\gamma - \frac{r_0}{v_v T})$ and decreasing otherwise. If $\theta_0 \in [\arccos\gamma, \arccos(\gamma - \frac{r_0}{v_v T})]$, $\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0} > 0$. From (A.4), we will have $\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0} < 0$.

When $\theta_0 \in (\arccos(\gamma - \frac{r_0}{v_v T}), \pi)$, $\frac{d\theta(T)}{d\theta_0} < 0$. Rewrite the relationship between $r(T)$ and $\theta(T)$ as following:

$$T = \frac{r_0(\gamma + \cos\theta_0) - r(T)(\gamma + \cos\theta(T))}{v_v(\gamma^2 - 1)},$$

or

$$v_v T(\gamma^2 - 1) = r_0(\gamma + \cos\theta_0) - r(T)(\gamma + \cos\theta(T)) \tag{A.5}$$

Taking the derivative with respect to θ_0 on both sides,

$$0 = -r_0 \sin\theta_0 - (\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0}(\gamma + \cos\theta(T)) - r(T)\sin\theta(T)\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0}).$$

Therefore,

$$\frac{\mathrm{d}r(T)}{\mathrm{d}\theta_0}(\gamma + \cos\theta(T)) = -r_0 \sin\theta_0 + r(T)\sin\theta(T)\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0}.$$

Since $\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0} < 0$, $-r_0 \sin\theta_0 + r(T)\sin\theta(T)\frac{\mathrm{d}\theta(T)}{\mathrm{d}\theta_0} <$. Besides, we assume that $\cos\theta(T) > \gamma > 0$. Hence $\gamma + \cos\theta(T) > 0$ and $\frac{dr(T)}{d\theta_0} < 0$.

Therefore, under the both two scenarios, we have proved the result in Proposition 2.4.

A.7 *Proof of Lemma 2.4*

To simplify the notation, let $r(\tau)$ and $\theta(\tau)$ denote the relative distance and LOS angle at τ. And $r(T)$, $\theta(T)$ denote the relative distance and LOS angle at T when two-turn policy $(\tau, \alpha_1, \alpha_2)$ is applied. In the first stage where $t \in [0, \tau]$, the dynamic process is same as one-turn policy. Thus $\frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}\theta(\tau)}{\mathrm{d}\tau}$ can be obtained from equations (2.7) and (2.8):

$$\begin{aligned} \frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau} &= -v_v(\gamma - \cos(\theta(\tau) - \alpha_1)) \\ \frac{\mathrm{d}\theta(\tau)}{\mathrm{d}\tau} &= -\frac{v_v \sin(\theta(\tau) - \alpha_1)}{r(\tau)} \end{aligned} \tag{A.6}$$

Now given $r(\tau)$ and $\theta(\tau)$, the derivatives of $\theta(T)$ and $r(T)$ on τ will be as following:

$$\begin{aligned}\frac{\tan^{\gamma}\frac{\theta(T)-\alpha_2}{2}}{\sin^2(\theta(T)-\alpha_2)}\frac{\mathrm{d}\theta(T)}{\mathrm{d}\tau} =& \frac{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}{\sin^2(\theta(\tau)-\alpha_2)}\frac{\mathrm{d}\theta(\tau)}{\mathrm{d}\tau} - \frac{v_v r(\tau)-v_1(T-\tau)}{r(\tau)^2}\frac{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}{\sin(\theta(\tau)-\alpha_2)} \\ &- \frac{v_v(T-\tau)}{r(\tau)}\frac{(\gamma-\cos(\theta(\tau)-\alpha_2))\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}{\sin^2(\theta(\tau)-\alpha_2)}\frac{\mathrm{d}\theta(\tau)}{\mathrm{d}\tau}\end{aligned} \tag{A.7}$$

$$\begin{aligned}\frac{\mathrm{d}r(T)}{\mathrm{d}\tau} =& \frac{\sin(\theta(\tau)-\alpha_2)}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\frac{\tan^{\gamma}\frac{\theta(T)-\alpha_2}{2}}{\sin(\theta(T)-\alpha_2)}\frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau} + r(\tau)\frac{\tan^{\gamma}\frac{\theta(T)-\alpha_2}{2}}{\sin(\theta(T)-\alpha_2)}\frac{\cos(\theta(\tau)-\alpha_2)-\gamma}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\frac{\mathrm{d}\theta(\tau)}{\mathrm{d}\tau} \\ &+ r(\tau)\frac{\sin(\theta(\tau)-\alpha_2)}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\frac{(\gamma-\cos(\theta(T)-\alpha_2))\tan^{\gamma}\frac{\theta(T)-\alpha_2}{2}}{\sin^2(\theta(T)-\alpha_2)}\frac{\mathrm{d}\theta(T)}{\mathrm{d}\tau}]\end{aligned} \tag{A.8}$$

Substituting equations (A.6) and (A.7) into equation (A.8),

$$\begin{aligned}&\frac{\mathrm{d}r(T)}{\mathrm{d}\tau} \\ =& \frac{v_v}{r(\tau)\sin(\theta(\tau)-\alpha_2)}\Big[r(T)\sin(\theta(\tau)-\alpha_1)[\gamma-\cos(\theta(\tau)-\alpha_2)] \\ &- r(T)\sin(\theta(\tau)-\alpha_2)[\gamma-\cos(\theta(\tau)-\alpha_1)] \\ &- r(\tau)\sin(\theta(\tau)-\alpha_1)[\gamma-\cos(\theta(T)-\alpha_2)] \\ &+ r(\tau)\sin(\theta(\tau)-\alpha_2)[\gamma-\cos(\theta(T)-\alpha_2)] \\ &- v_v(T-\tau)\sin(\theta(\tau)-\alpha_2)[\gamma-\cos(\theta(T)-\alpha_2)][\gamma-\cos(\theta(\tau)-\alpha_1)] \\ &+ v_v(T-\tau)\sin(\theta(\tau)-\alpha_1)[\gamma-\cos(\theta(T)-\alpha_2)][\gamma-\cos(\theta(\tau)-\alpha_2)]\Big] \\ =& \frac{v_v}{r(\tau)\sin(\theta(\tau)-\alpha_2)}\Big[K_1\sin(\theta_1-\alpha_2) - K_2\sin(\theta_1-\alpha_1)\Big]\end{aligned}$$

where $K_1 = r(\tau)(\gamma-\cos(\theta(T)-\alpha_2)) - v_v(T-\tau)(\gamma-\cos(\theta(\tau)-\alpha_1))(\gamma-\cos(\theta(T)-\alpha_2)) - r(T)(\gamma-\cos(\theta(\tau)-\alpha_1))$, $K_2 = r(\tau)(\gamma-\cos(\theta(T)-\alpha_2)) - v_v(T-\tau)(\gamma-\cos(\theta(\tau)-\alpha_2))(\gamma-\cos(\theta(T)-\alpha_2)) - r(T)(\gamma-\cos(\theta(\tau)-\alpha_2))$

As we assumed that $\alpha_2 \in [\theta(\tau)-\pi, \theta(\tau))$, $\theta(\tau)-\alpha_2 \in [0,\pi]$, hence $\sin(\theta(\tau)-\alpha_2) \geq 0$.

Consider the part $Ftriangleq = \dfrac{K_1\sin(\theta(\tau)-\alpha_2) - K_2\sin(\theta(\tau)-\alpha_1)}{r(\tau)}$.

$$\begin{aligned}F = 2\sin\frac{\alpha_1-\alpha_2}{2}\Big[&(A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2))\cos\frac{\theta(\tau)-\alpha_1}{2}\cos\frac{\theta(\tau)-\alpha_2}{2} \\ &+(A-\gamma+A\gamma+\cos(\theta(T)-\alpha_2))\sin\frac{\theta(\tau)-\alpha_1}{2}\sin\frac{\theta(\tau)-\alpha_2}{2}\Big]\end{aligned} \tag{A.9}$$

where $A = \dfrac{r(T)+v_v(T-\tau)(\gamma-\cos(\theta(T)-\alpha_2))}{r(\tau)}$. As we assume that $\gamma-\cos(\theta(T)-\alpha_2) > 0$, it implies that $0 < A < 1$.

Then what we need to prove is that both $A-\gamma+A\gamma+\cos(\theta(T)-\alpha_2)$ and $A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2)$ are nonnegative.

For $A-\gamma+A\gamma+\cos(\theta(T)-\alpha_2)$, we have

$$\begin{aligned}
&A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2)\\
&=(1-A)\gamma-\cos(\theta(T)-\alpha_2)+A\cos(\theta(T)-\alpha_2)+A(1-\cos(\theta(T)-\alpha_2))\\
&=(1-A)(\gamma-\cos(\theta(T)-\alpha_2))+A(1-\cos(\theta(T)-\alpha_2))>0
\end{aligned} \tag{A.10}$$

Now that $0<A<1$, $\cos(\theta(T)-\alpha_2)<1$ and $\gamma-\cos(\theta(T)-\alpha_2)>0$, we can conclude that $A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2)>0$.

On the other hand, we can obtain

$$\begin{aligned}
&A-\gamma+A\gamma+\cos(\theta(T)-\alpha_2)\\
&=(1+\gamma)\frac{r(T)+v_v(T-\tau)(\gamma-\cos(\theta(T)-\alpha_2))}{r(\tau)}+(\cos(\theta(T)-\alpha_2)-\gamma)\\
&=(\gamma+1)\frac{r(T)+v_v(T-\tau)(\gamma-\cos(\theta(T)-\alpha_2))}{r(\tau)}-(\gamma-\cos(\theta(T)-\alpha_2))\\
&=\frac{(\gamma+1)r(T)+(v_v(T-\tau)(\gamma+1)-r(\tau))(\gamma-\cos(\theta(T)-\alpha_2))}{r(\tau)}\\
&=\frac{(1+\cos(\theta(T)-\alpha_2))r(T)+(r(T)+v_v(T-\tau)(\gamma+1)-r(\tau))(\gamma-\cos(\theta(T)-\alpha_2))}{r(\tau)}
\end{aligned} \tag{A.11}$$

As $r(T)+v_v(T-\tau)(\gamma+1)-r(\tau)\geq r(T)+v_v(T-\tau)(\gamma-\cos(\theta(T)-\alpha_2))-r(\tau)>0$ and $\gamma-\cos(\theta(T)-\alpha_2)>0$, $A-\gamma+A\gamma+\cos(\theta(T)-\alpha_2)>0$.

Above all, $\dfrac{\mathrm{d}r(T)}{\mathrm{d}\tau}>0$ when $\alpha_1\geq\alpha_2$ and <0, otherwise.

A.8 Proof of Lemma 2.5

For the minimum relative distance in the first stage, the result is actually same as in the direct heading case. The minimum distance is independent with the turn time.

For the minimum relative distance in the second stage, the minimum relative distance occurs when $\theta_2(t)-\alpha_2=\arccos\gamma$. Then

$$\bar{r}(\gamma)=\frac{r(\tau)\sin(\theta(\tau)-\alpha_2)}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\frac{\tan^{\gamma}\frac{\arccos\gamma}{2}}{\sin(\arccos\gamma)}$$

Let $c \triangleq \frac{\tan^{\gamma}\frac{\arccos\gamma}{2}}{\sin(\arccos\gamma})$,

$$
\begin{aligned}
&\frac{\mathrm{d}\bar{r}(\gamma)}{d\tau}\\
=&c\frac{\sin(\theta(\tau)-\alpha_2)}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\frac{dr(\tau)}{d\tau}+cr(\tau)\frac{\cos(\theta(\tau)-\alpha_2)-\gamma}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\frac{d\theta(\tau)}{d\tau}\\
=&c\Big[\frac{\sin(\theta_1-\alpha_2)}{\tan^{\gamma}\frac{\theta_1-\alpha_2}{2}}v_v(\cos(\theta(\tau)-\alpha_1)-\gamma)+r(\tau)\frac{\cos(\theta(\tau)-\alpha_2)-\gamma}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}v_v\frac{\sin(\alpha_1-\theta_1)}{r(\tau)}\Big]\\
=&\frac{v_v c}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\Big[(\gamma-\cos(\theta(\tau)-\alpha_2))\sin(\theta(\tau)-\alpha_1)-(\gamma-\cos(\theta(\tau)-\alpha_1))\sin(\theta(\tau)-\alpha_2)\Big]\\
=&\frac{v_v c}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\Big[2\sin\frac{\alpha_1-\alpha_2}{2}(\cos\frac{\alpha_1-\alpha_2}{2}-\gamma\cos(\theta(\tau)-\frac{\alpha_1+\alpha_2}{2}))\Big]
\end{aligned}
$$

Now we only need to prove that $\cos\frac{\alpha_1-\alpha_2}{2}-\gamma\cos(\theta(\tau)-\frac{\alpha_1+\alpha_2}{2})\geq 0$.

$$
\begin{aligned}
&\cos\frac{\alpha_1-\alpha_2}{2}-\gamma\cos(\theta(\tau)-\frac{\alpha_1+\alpha_2}{2})\\
=&\frac{v_v c}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\Big[\cos(\frac{\theta(\tau)-\alpha_2}{2}-\frac{\theta(\tau)-\alpha_1}{2})-\gamma\cos(\frac{\theta(\tau)-\alpha_1}{2}+\frac{\theta(\tau)-\alpha_2}{2})\Big]\\
=&\frac{v_v c}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\Big[(1-\gamma)\cos\frac{\theta(\tau)-\alpha_1}{2}\cos\frac{\theta(\tau)-\alpha_2}{2}+(1+\gamma)\sin\frac{\theta(\tau)-\alpha_1}{2}\sin\frac{\theta(\tau)-\alpha_2}{2}\Big]
\end{aligned}
$$

As all items are nonnegative, we can conclude $\cos\frac{\alpha_1-\alpha_2}{2}-\gamma\cos(\theta(\tau)-\frac{\alpha_1+\alpha_2}{2})\geq 0$, and thus $\frac{\mathrm{d}\bar{r}(\gamma)}{d\tau}\geq 0$ when $\alpha_1>\alpha_2$ and $\frac{\mathrm{d}\bar{r}(\gamma)}{d\tau}<0$ when $\alpha_1<\alpha_2$.

A.9 Proof of Proposition 2.5

When $\gamma>1$, $r(t)$ is strictly decreasing during $t\in[0,T]$, and the sufficient and necessary condition for a two-turn policy being feasible is $r(T;\tau,\alpha_1,\alpha_2)\geq R$. With Lemma 2.4, we can directly conclude the result.

When $\gamma<1$, $r(t)$ may be strictly decreasing during $t\in[0,T]$, or first decreasing and then increasing. The condition is divided into two scenarios.

If $\gamma-\cos(\theta(T;\tau,\alpha_1,\alpha_2)-\alpha_2)\geq 0$, $r(t)$ will be decreasing on $t\in[\tau,T]$. If $\alpha_1>\alpha^*>\alpha_2$, we have $r(t)>R$ during $t\in[0,\tau]$ for any turn time τ. Therefore, the sufficient and necessary condition is still $r(T;\tau,\alpha_1,\alpha_2)\geq R$. The result holds same as the case $\gamma>1$. If $\alpha_1<\alpha^*<\alpha_2$, $\gamma-\cos(\theta(\tau;\tau,\alpha_1,\alpha_2)-\alpha_1)\leq 0$ as well, which implies $r(t)$ is decreasing during $t\in[0,T]$ and thus the sufficient and necessary condition will be $r(T;\tau,\alpha_1,\alpha_2)\geq R$.

With Lemma 2.4, we can conclude that the vessel is safe before a specific τ and the result holds.

On the other hand, if $\gamma - \cos(\theta(T;\tau,\alpha_1,\alpha_2) - \alpha_2) < 0$, which implies $r(T;\tau,\alpha_1,\alpha_2)$ is not critical for the vessel to be safe. If $\alpha_1 > \alpha^* > \alpha_2$, the vessel is always safe during $t \in [0,\tau]$. Now that (τ,α_1,α_2) is feasible, $(\tau',\alpha_1,\alpha_2)$ is feasible as well since $\bar{r}(\gamma,\tau,\alpha_1,\alpha_2)$ of the second stage is increasing on τ. For the case of $\alpha_1 < \alpha^* < \alpha_2$, $r(t)$ can only be decreasing on $t \in [0,\tau]$. Therefore, $r(\tau') > r(\tau)$ if $\tau' < \tau$. Meanwhile, the minimum relative distance $\bar{r}(\gamma;\tau,\alpha_1,\alpha_2)$ during the second stage is decreasing on τ. If $\bar{r}(\gamma;\tau,\alpha_1,\alpha_2)$ does not exist, it implies that $r(t)$ is increasing during $t \in [\tau,T]$ and the condition only depends on the first stage and thus the result holds as $\alpha_1 < \alpha^*$.

Therefore, the result holds.

APPENDIX B

TECHNICAL SUPPORTS FOR CHAPTER IV

B.1 Proof of Theorem 4.1

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To help prove the theorem, we define a new value function $\hat{v}_i(Q_{i+1})$ where

$$\hat{v}_i(Q_{i+1}) = -p_i^s Q_{i+1} + \mathbf{E}_{R_{i+1},S_{i+1}} \pi_{i+1}(Q_{i+1}, R_{i+1}, S_{i+1})$$

The proof of the theorem is equivalent to proving the value function $\hat{v}_i(Q_{i+1})$ is a concave function on $Q_{i+1} \in \{\sum_{j=i+1}^{n} \bar{R}_j, \cdots, +\infty\}$.

We will prove this by induction.

When $i = n$, $\hat{v}_n(Q_{n+1}) = -p_n^s Q_{n+1} + \mathbf{E}_{R_{n+1},S_{n+1}} \Pi_{n+1}(Q_{n+1}, R_{n+1}, S_{n+1})$. Since the feeder vessel will return to the hub port after port n to unload all the laden containers, $\Pi_{n+1}(Q_{n+1}, R_{n+1}, S_{n+1}) = 0$. It implies that $\hat{v}_n(Q_{n+1}) = -p_n Q_{n+1}$, which is a concave function

Now we assume that $\hat{v}_{i+1}(Q_{i+2})$ is a concave function on $Q_{i+2} \in \{\sum_{j=i+2}^{n} \bar{R}_j, \cdots, +\infty\}$. We will prove that $v_i(\hat{Q}_{i+1})$ is a concave function. Since

$$\begin{aligned}\hat{v}_i(Q_{i+1}) &= -p_i^s Q_{i+1} + \mathbf{E}_{R_{i+1},S_{i+1}} \pi_{i+1}(Q_{i+1}, R_{i+1}, S_{i+1}) \\ &= \mathbf{E}_{R_{i+1},S_{i+1}} \{-p_i^s Q_{i+1} + \pi_{i+1}(Q_{i+1}, R_{i+1}, S_{i+1})\},\end{aligned}$$

what we need to prove is now $\Pi_{i+1}(Q_{i+1}, R_{i+1}, S_{i+1})$ is a concave function on Q_{i+1} for any given $R_{i+1} = r_{i+1}$ and $S_{i+1} = s_{i+1}$. The maximal expected revenue at port $i+1$ with given Q_{i+1}, r_{i+1} and s_{i+1} is

$$\begin{aligned}\pi_{i+1}(Q_{i+1}, r_{i+1}, s_{i+1}) &= \max_{Q_{i+2}} \quad v_i(Q_{i+1}, Q_{i+2}, r_{i+2}, s_{i+2}) \\ &= \max_{Q_{i+2}} \quad p_{i+1}^r r_{i+1} + p_{i+1}^s Q_{i+1} + \hat{v}_{i+1}(Q_{i+2})\end{aligned} \tag{B.1}$$

where $Q_{i+2} \in \{Q_{i+1} - r_{i+1} - s_{i+1}, \cdots, Q_{i+1} - r_{i+1}\}$.

As $\hat{v}_{i+1}(Q_{i+2})$ is concave, the optimal serving policy is adoptable for port $i+1$. Assume Q^*_{i+2} maximizes $\hat{v}_{i+1}(Q_{i+2})$, then

$$\pi_{i+1}(Q_{i+1}, r_{i+1}, s_{i+1})$$
$$= \begin{cases} p^r_{i+1} r_{i+1} + p^s_{i+1} Q_{i+1} + \hat{v}_{i+1}(Q_{i+1} - r_{i+1}), & \text{if } Q_{i+1} - r_{i+1} < Q^*_{i+2} \\ p^r_{i+1} r_{i+1} + p^s_{i+1} Q_{i+1} + \hat{v}_{i+1}(Q^*_{i+2}), & \text{if } Q^*_{i+2} \le Q_{i+1} - r_{i+1} \le Q^*_{i+2} + s_{i+1} \\ p^r_{i+1} r_{i+1} + p^s_{i+1} Q_{i+1} + \hat{v}_{i+1}(Q_{i+1} - r_{i+1} - s_{i+1}), & \text{if } Q_{i+1} - r_{i+1} > Q^*_{i+2} + s_{i+1} \end{cases} \tag{B.2}$$

On each interval, $\pi_{i+1}(Q_{i+1}, r_{i+1}, s_{i+1})$ is concave. Now look at the forward difference of $\pi_{i+1}(Q_{i+1}, r_{i+1}, s_{i+1})$ when $Q^1_{i+1} = Q^*_{i+2} + r_{i+1}$ and $Q^2_{i+1} = Q^*_{i+2} + r_{i+1} + s_{i+1}$. We have

$$\begin{cases} \pi_{i+1}(Q^1_{i+1}, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^1_{i+1} - 1, r_{i+1}, s_{i+1}) = p^s_{i+1} + \hat{v}_{i+1}(Q^*_{i+2}) - \hat{v}_{i+1}(Q^*_{i+2} - 1) \\ \pi_{i+1}(Q^1_{i+1} + 1, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^1_{i+1}, r_{i+1}, s_{i+1}) = p^s_{i+1} \end{cases}$$
$$\begin{cases} \pi_{i+1}(Q^2_{i+1}, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^2_{i+1} - 1, r_{i+1}, s_{i+1}) = p^s_{i+1} \\ \pi_{i+1}(Q^2_{i+1} + 1, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^2_{i+1}, r_{i+1}, s_{i+1}) = p^s_{i+1} + \hat{v}_{i+1}(Q^*_{i+2} + 1) - \hat{v}_{i+1}(Q^*_{i+2}) \end{cases}$$

As $\hat{v}_{i+1}(Q_{i+2})$ is concave, it means

$$\hat{v}_{i+1}(Q^*_{i+2}) - \hat{v}_{i+1}(Q^*_{i+2} - 1) \ge 0 \ge \hat{v}_{i+1}(Q^*_{i+2} + 1) - \hat{v}_{i+1}(Q^*_{i+2})$$

Hence, $\pi_{i+1}(Q^1_{i+1}, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^1_{i+1} - 1, r_{i+1}, s_{i+1}) \ge \pi_{i+1}(Q^1_{i+1} + 1, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^1_{i+1}, r_{i+1}, s_{i+1})$, and $\pi_{i+1}(Q^2_{i+1}, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^2_{i+1} - 1, r_{i+1}, s_{i+1}) \ge \pi_{i+1}(Q^2_{i+1} + 1, r_{i+1}, s_{i+1}) - \pi_{i+1}(Q^2_{i+1}, r_{i+1}, s_{i+1})$.

Therefore, $\pi_{i+1}(Q_{i+1}, r_{i+1}, s_{i+1})$ is a concave function on Q_{i+1} and so does $\hat{v}_i(Q_{i+1})$.

B.2 Proof of Theorem 4.2

We still use $\hat{v}_i(Q_{i+1})$ defined in above section to prove the result, and

$$\hat{v}_i(Q_{i+1}) = -p^s_i Q_{i+1} + \mathbf{E}_{R_{i+1}, S_{i+1}} \pi_{i+1}(Q_{i+1}, R_{i+1}, S_{i+1})$$
$$\mathbf{E}_{R_{i+1}, S_{i+1}} \{-p^s_i Q_{i+1} + \pi_{i+1}(Q_{i+1}, R_{i+1}, S_{i+1})\}$$

For any given $R_{i+1} = r_{i+1}$ and $S_{i+1} = s_{i+1}$, by substituting equation B.2 into $\hat{v}_i(Q_{i+1})$, we have

$$g(Q_{i+1}) \triangleq -p_i^s Q_{i+1} + \pi_{i+1}(Q_{i+1}, r_{i+1}, s_{i+1}) =$$

$$\begin{cases} p_{i+1}^r r_{i+1} + (p_{i+1}^s - p_i^s)Q_{i+1} + \hat{v}_{i+1}(Q_{i+1} - r_{i+1}), & \text{if } Q_{i+1} - r_{i+1} < Q_{i+2}^* \\ p_{i+1}^r r_{i+1} + (p_{i+1}^s - p_i^s)Q_{i+1} + \hat{v}_{i+1}(Q_{i+2}^*), & \text{if } Q_{i+2}^* \leq Q_{i+1} - r_{i+1} \leq Q_{i+2}^* + s_{i+1} \\ p_{i+1}^r r_{i+1} + (p_{i+1}^s - p_i^s)Q_{i+1} + \hat{v}_{i+1}(Q_{i+1} - r_{i+1} - s_{i+1}), & \text{if } Q_{i+1} - r_{i+1} > Q_{i+2}^* + s_{i+1} \end{cases}$$

When $p_i^s < p_{i+1}^s$, $g(Q_{i+1})$ is increasing on $Q_{i+1} \in \{Q_{i+2}^* + r_{i+1}, \cdots, Q_{i+2}^* + r_{i+1} + s_{i+1}\}$. Now that $g(Q_{i+1})$ is concave, we will have $Q_{i+1}^* > Q_{i+2}^* + r_i + s_i$. As the result holds for any r_{i+1} and s_{i+1}, we can conclude that $Q_{i+1}^* > Q_{i+2}^*$.

When $p_i^s > p_{i+1}^s$, $g(Q_{i+1})$ is decreasing on $Q_{i+1} \in \{Q_{i+2}^* + r_{i+1}, \cdots, Q_{i+2}^* + r_{i+1} + s_{i+1}\}$. Therefore, $Q_{i+1}^* < Q_{i+2}^* + r_{i+1}$ for any r_{i+1}. We can thus conclude that $Q_{i+1}^* < Q_{i+2}^* + \bar{R}_{i+1}$.

B.3 Proof of Lemma 4.2

According to the definition, for any given $x_1, x_2 \in S_1$, there exist y_1 and $y_2 \in S_2$ such that

$$f(x_1, y_1) = g(x_1) \text{ and } f(x_2, y_2) = g(x_2)$$

To prove that $g(x)$ is a discretely concave function, we need to show that for any $\alpha \in (0, 1)$, it holds that

$$\max_{u \in N(z)} g(u) \geq \alpha g(x_1) + (1 - \alpha)g(x_2)$$

where $z = \alpha x_1 + (1 - \alpha)x_2$.

Note that

$$\begin{aligned} \alpha g(x_1) + (1-\alpha)g(x_2) &= \alpha f(x_1, y_1) + (1-\alpha)f(x_2, y_2) \\ &\leq \max_{u_1 \in N(\alpha x_1 + (1-\alpha)x_2), u_2 \in N(\alpha y_1 + (1-\alpha)y_2)} f(u_1, u_2) \\ &\leq \max_{u_1 \in N(\alpha x_1 + (1-\alpha)x_2), u_2 \in S_2} f(u_1, u_2) \\ &= \max_{u_1 \in N(\alpha x_1 + (1-\alpha)x_2)} \max_{u_2 \in S_2} f(u_1, u_2) \\ &= \max_{u_1 \in N(\alpha x_1 + (1-\alpha)x_2)} g(u_1). \end{aligned} \quad \text{(B.3)}$$

Hence, $g(x)$ is a discretely concave function according to the definition.

B.4 Proof of Theorem 4.3

We will prove the result by induction.

When $i = n$, we have

$$\pi_n(Q_n^p, Q_n^d, r_n, s_n, d_n, e_n) = (p_n^r - p_n^s)r_n + (p_n^d - p_n^e)d_n + p_n^s\hat{Q}_n^p + p_n^e\hat{Q}_n^d$$

$$v_n(Q_n^p, Q_n^d, Q_{n+1}^p, Q_{n+1}^d, r_n, s_n, d_n, e_)$$

$$=(p_n^r - p_n^s)r_n + (p_n^d - p_n^e)d_n + p_n^s Q_n^p + (p_n^s + p_n^e)Q_n^d - p_n^s Q_n^p - (p_n^s + p_n^e)Q_n^d$$

where $\hat{Q}_n^d = \min\{Q_n^d, d_n + e_n\}$ and $\hat{Q}_n^p = \min\{Q_n^p + \hat{Q}_n^d, r_n + s_n\}$. We are able to check the discretely concavity by definition.

Now we assume that the result holds for $i+1$, which means $v_{i+1}(Q_{i+1}^p, Q_{i+1}^d, Q_{i+2}^p, Q_{i+2}^d, r_{i+1}, s_{i+1}, d_{i+1}, e_{i+1})$ and $\pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d, r_{i+1}, s_{i+1}, d_{i+1}, e_{i+1})$ are discretely concave function for any given $(r_{i+1}, s_{i+1}, d_{i+1}, e_{i+1})$.

Then

$$v_i(Q_i^p, Q_i^d, Q_{i+1}^p, Q_{i+1}^d, r_i, s_i, d_i, e_i)$$

$$=(p_i^r - p_i^s)r_i + (p_i^d - p_i^e)d_i + p_i^s Q_i^p + (p_i^s + p_i^e)Q_i^d - p_i^s Q_{i+1}^p - (p_i^s + p_i^e)Q_{i+1}^d + h_i(Q_{i+1}^p, Q_{i+1}^d)$$

is discretely concave function for any (r_i, s_i, d_i, e_i).

From Lemma 4.2, we have that $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ is discretely concave function as well.

B.5 Proof of Lemma 4.3

Firstly, we define the following function.

$$\hat{v}_i(Q_{i+1}^p, Q_{i+1}^d) = -p_i^s Q_{i+1}^p - (p_i^s + p_i^e)Q_{i+1}^d + h_i(Q_{i+1}^p, Q_{i+1}^d).$$

Then $\hat{v}_i(Q_{i+1}^p, Q_{i+1}^d)$ is a discretely concave function on (Q_{i+1}^p, Q_{i+1}^d).

To prove the result, it suffices to prove that $\hat{v}_i(Q_{i+1}^p + 1, Q_{i+1}^d - 1) - \hat{v}_i(Q_{i+1}^p, Q_{i+1}^d) \leq \hat{v}_i(Q_{i+1}^p, Q_{i+1}^d) - \hat{v}_i(Q_{i+1}^p - 1, Q_{i+1}^d + 1)$, which is equivalent to

$$\hat{v}_i(Q_{i+1}^p + 1, Q_{i+1}^d - 1) + \hat{v}_i(Q_{i+1}^p - 1, Q_{i+1}^d + 1) \leq 2\hat{v}_i(Q_{i+1}^p, Q_{i+1}^d).$$

Since $\hat{v}_i(Q_{i+1}^p, Q_{i+1}^d)$ is discretely concave on (Q_{i+1}^p, Q_{i+1}^d), we have

$$\frac{1}{2}\hat{v}_i(Q_{i+1}^p + 1, Q_{i+1}^d - 1) + \frac{1}{2}\hat{v}_i(Q_{i+1}^p - 1, Q_{i+1}^d + 1) \leq \max_{u_1 \in N(Q_{i+1}^p), u_2 \in N(Q_{i+1}^d)} \hat{v}_i(u_1, u_2)$$

while $\max_{u_1 \in N(Q^p_{i+1}), u_2 \in N(Q^d_{i+1})} \hat{v}_i(u_1, u_2) = \hat{v}_i(Q^p_{i+1} + 1, Q^d_{i+1} - 1)$. Hence, $\hat{v}_i(Q^p_{i+1}, Q^d_{i+1})$ is concave on Q^p_{i+1}.

We can conclude $\hat{v}_i(Q^p_{i+1}, Q^d_{i+1})$ is concave on Q^d_{i+1} with the same method.

B.6 Proof of Lemma 4.4

We slightly change the notation here by using $\pi_i(Q^p_i, Q^d_i, \hat{c})$ for simple presentation.

Here $\pi_i(Q^p_i, Q^d_i, \hat{c}) = \max_{Q^p_{i+1}} (p^r_i - p^s_i) r_i + (p^d_i - p^e_i) d_i + p^s_i Q^p_i + (p^s_i + p^e_i) Q^d_i + \hat{v}_i(Q^p_{i+1}, Q^d_{i+1})$, and let $(Q^{p*}_{i+1}(\hat{c}), Q^{d*}_{i+1}(\hat{c}))$ denote the optimal solution when $Q^p_{i+1} + Q^d_{i+1} = \hat{c}$ (note that $(Q^{p*}_{i+1}, Q^{d*}_{i+1})$ can be different from $(Q^p_{i+1}(\hat{c}), Q^d_{i+1}(\hat{c}))$).

There are four cases that we need to consider:

i) $Q^{p*}_{i+1}(\hat{c}+1) = Q^{p*}_{i+1}(\hat{c}) \Rightarrow Q^{d*}_{i+1}(\hat{c}+1) = Q^{d*}_{i+1}(\hat{c})+1$, $Q^{p*}_{i+1}(\hat{c}-1) = Q^{p*}_{i+1}(\hat{c}) \Rightarrow Q^{d*}_{i+1}(\hat{c}-1) = Q^{d*}_{i+1}(\hat{c}) - 1$.

Now the forward difference will be

$\pi_i(Q^p_i, Q^d_i, \hat{c}+1) - \pi_i(Q^p_i, Q^d_i, \hat{c}) = -(p^s_i + p^e_i) + h_i(Q^{p*}_{i+1}(\hat{c}), Q^{d*}_{i+1}(\hat{c})+1) - h_i(Q^{p*}_{i+1}(\hat{c}), Q^{d*}_{i+1}(\hat{c}))$

and

$\pi_i(Q^p_i, Q^d_i, \hat{c}) - \pi_i(Q^p_i, Q^d_i, \hat{c}-1) = -(p^s_i + p^e_i) + h_i(Q^{p*}_{i+1}(\hat{c}), Q^{d*}_{i+1}(\hat{c})) - h_i(Q^{p*}_{i+1}(\hat{c}), Q^{d*}_{i+1}(\hat{c})-1)$

Hence $\pi_i(Q^p_i, Q^d_i, \hat{c}+1) - \pi_i(Q^p_i, Q^d_i, \hat{c}) \le \pi_i(Q^p_i, Q^d_i, \hat{c}) - \pi_i(Q^p_i, Q^d_i, \hat{c}-1)$ due to the concavity on Q^d_{i+1}.

ii) $Q^{p*}_{i+1}(\hat{c}+1) = Q^{p*}_{i+1}(\hat{c}) + 1 \Rightarrow Q^{d*}_{i+1}(\hat{c}+1) = Q^{d*}_{i+1}(\hat{c})$, $Q^{p*}_{i+1}(\hat{c}-1) = Q^{p*}_{i+1}(\hat{c}) - 1 \Rightarrow Q^{d*}_{i+1}(\hat{c}-1) = Q^{d*}_{i+1}(\hat{c})$.

Now the forward difference is

$\pi_i(Q^p_i, Q^d_i, \hat{c}+1) - \pi_i(Q^p_i, Q^d_i, \hat{c}) = -p^s_i + h_i(Q^{p*}_{i+1}(\hat{c})+1, Q^{d*}_{i+1}(\hat{c})) - h_i(Q^{p*}_{i+1}(\hat{c}), Q^{d*}_{i+1}(\hat{c}))$

and

$\pi_i(Q^p_i, Q^d_i, \hat{c}) - \pi_i(Q^p_i, Q^d_i, \hat{c}-1) = -p^s_i + h_i(Q^{p*}_{i+1}(\hat{c}), Q^{d*}_{i+1}(\hat{c})) - h_i(Q^{p*}_{i+1}(\hat{c})-1, Q^{d*}_{i+1}(\hat{c}))$

Hence $\pi_i(Q^p_i, Q^d_i, \hat{c}+1) - \pi_i(Q^p_i, Q^d_i, \hat{c}) \le \pi_i(Q^p_i, Q^d_i, \hat{c}) - \pi_i(Q^p_i, Q^d_i, \hat{c}-1)$ due to the concavity on Q^d_{i+1}.

iii) $Q^{p*}_{i+1}(\hat{c}+1) = Q^{p*}_{i+1}(\hat{c}) \Rightarrow Q^{d*}_{i+1}(\hat{c}+1) = Q^{d*}_{i+1}(\hat{c}) + 1$, $Q^{p*}_{i+1}(\hat{c}-1) = Q^{p*}_{i+1}(\hat{c}) - 1 \Rightarrow Q^{d*}_{i+1}(\hat{c}-1) = Q^{d*}_{i+1}(\hat{c})$.

Now the forward difference is

$\pi_i(Q_i^p, Q_i^d, \hat{c}+1) - \pi_i(Q_i^p, Q_i^d, \hat{c}) = -(p_i^s + p_i^e) + h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c})+1) - h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}))$

and

$\pi_i(Q_i^p, Q_i^d, \hat{c}) - \pi_i(Q_i^p, Q_i^d, \hat{c}-1) = -p_i^s + h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c})) - h_i(Q_{i+1}^{p*}(\hat{c}) - 1, Q_{i+1}^{d*}(\hat{c}))$

iv) $Q_{i+1}^{p*}(\hat{c}+1) = Q_{i+1}^{p*}(\hat{c}) + 1 \Rightarrow Q_{i+1}^{d*}(\hat{c}+;1) = Q_{i+1}^{d*}(\hat{c})$, $Q_{i+1}^{p*}(\hat{c}-1) = Q_{i+1}^{p*}(\hat{c}) \Rightarrow Q_{i+1}^{d*}(\hat{c}-1) = Q_{i+1}^{d*}(\hat{c}) - 1$.

Now the forward difference is

$\pi_i(Q_i^p, Q_i^d, \hat{c}+1) - \pi_i(Q_i^p, Q_i^d, \hat{c}) = -p_i^s + h_i(Q_{i+1}^{p*}(\hat{c}) + 1, Q_{i+1}^{d*}(\hat{c})) - h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}))$

and

$\pi_i(Q_i^p, Q_i^d, \hat{c}) - \pi_i(Q_i^p, Q_i^d, \hat{c}-1) = -(p_i^s + p_i^e) + h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(c)) - h_i(Q_{i+1}^{p*}(c), Q_{i+1}^{d*}(\hat{c}) - 1)$

If we assume that $\pi_i(Q_i^p, Q_i^d, \hat{c}+1) - \pi_i(Q_i^p, Q_i^d, \hat{c}) > \pi_i(Q_i^p, Q_i^d, \hat{c}) - \pi_i(Q_i^p, Q_i^d, \hat{c}-1)$, we will have

$$\begin{aligned} & h_i(Q_{i+1}^{p*}(\hat{c}) + 1, Q_{i+1}^{d*}(\hat{c})) - h_i(Q_{i+1}^{p*}(\hat{c}s), Q_{i+1}^{d*}(\hat{c})) \\ > & - p_i^e + h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c})) - h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}) - 1) \end{aligned}$$

The submodularity implies that

$$\begin{aligned} & h_i(Q_{i+1}^{p*}(\hat{c}) + 1, Q_{i+1}^{d*}(\hat{c}) - 1) - h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}) - 1) \\ \geq & h_i(Q_{i+1}^{p*}(\hat{c}) + 1, Q_{i+1}^{d*}(\hat{c})) - h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c})) \end{aligned}$$

Therefore,

$$\begin{aligned} & (Q_{i+1}^{p*}(\hat{c}) + 1, Q_{i+1}^{d*}(\hat{c}) - 1) - h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}) - 1) \\ > & - p_i^e + h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c})) - h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}) - 1) \end{aligned}$$

which means

$$p_i^e + h_i(Q_{i+1}^{p*}(\hat{c}) + 1, Q_{i+1}^{d*}(\hat{c}) - 1) + h_i(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c})) > 0.$$

This inequality means that $(Q_{i+1}^{p*}(\hat{c}) + 1, Q_{i+1}^{d*}(\hat{c}) - 1)$ is better than $(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}))$ given $Q_{i+1}^p + Q_{i+1}^d = \hat{c}$, which is a contradiction with the fact $(Q_{i+1}^{p*}(\hat{c}), Q_{i+1}^{d*}(\hat{c}))$ is optimal.

Above all, we have the inequality that $\pi_i(Q_i^p, Q_i^d, \hat{c}+1) - \pi_i(Q_i^p, Q_i^d, \hat{c}) \leq \pi_i(Q_i^p, Q_i^d, \hat{c}) - \pi_i(Q_i^p, Q_i^d, \hat{c}-1)$ for any $\hat{c}$ and any r_i, s_i, d_i, e_i. And thus $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ is concave on $\hat{c}$ for any r_i, s_i, d_i, e_i.

B.7 Proof of Theorem 4.5

This can be concluded directly from the concavity on $\hat{c}$ for given (Q_i^p, Q_i^d) and the optimal solution is c^*. For given (Q_i^p, Q_i^d), $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ is increasing on $\{\cdots, c^*-1, c^*\}$ and decreasing on $\{c^*, c^*+1, \cdots\}$. Meanwhile, $\hat{c}$ is reduced only by collecting laden container. Therefore, when $\hat{c} \leq c^*$, we have $x_i^{s*} = 0$. On the other hand, if $\hat{c} \geq c^* + s_i$, then $x_i^{s*} = s_i$ if $Q_i^p + d_i + x_i^{e*} - r_i - s_i \geq \sum\limits_{j=i+1}^{n} \bar{R}_j$ and $x_i^{s*} = Q_i^p + d_i + x_i^{e*} - r_i - s_i - \sum\limits_{j=i+1}^{n} \bar{R}_j$ if $Q_i^p + d_i + x_i^{e*} - r_i - s_i \leq \sum\limits_{j=i+1}^{n} \bar{R}_j$. However, if $Q_i^p + d_i + x_i^{e*} - r_i - s_i \leq \sum\limits_{j=i+1}^{n} \bar{R}_j$, then x_i^{s*} can be increased by increasing x_i^{e*}, hence, we conclude that $x_i^{e*} = e_i$. Therefore, $x_i^{s*} = L_i \triangleq Q_i^p + d_i + e_i - r_i - \sum\limits_{j=i+1}^{n} \bar{R}_j$ when $Q_i^p + d_i + e_i - r_i - s_i \leq \sum\limits_{j=i+1}^{n} \bar{R}_j$. And $x_i^{s*} = s_i$ when $Q_i^p + d_i + e_i - r_i - s_i > \sum\limits_{j=i+1}^{n} \bar{R}_j$. After deciding x_i^{s*}, the optimal remaining capacities can be found by Lemma 4.4.

B.8 Proof of Lemma 4.5

Since $h_i(Q_{i+1}^p, Q_{i+1}^d)$ is a submodular function on (Q_{i+1}^p, Q_{i+1}^d), we can conclude $Q_{i+1}^p(k+1) \leq Q_{i+1}^p(k)$ directly due to the property of decreasing difference. Hence, we only need to discuss the case $Q_{i+1}^p(k+1) \geq Q_{i+1}^p(k) - 1$. This is equivalent to proving that

$$h_i(Q_{i+1}^p(k) - 1, k+1) - h_i(Q_{i+1}^p(k) - 2, k+1) > p_i^s.$$

Now that $Q_{i+1}^p(k)$ is the optimal solution given $Q_{i+1}^d = k$, we have

$$h_i(Q_{i+1}^p(k), k) - h_i(Q_{i+1}^p(k) - 1, k) > p_i^s.$$

Compare the two cases $Q_{i+1}^d = k$ and $Q_{i+1}^d = k+1$. There would be only two possibilities, either same delivery decision or one more delivery in case $k+1$.

For the case of same delivery decision, $h_i(Q_{i+1}^p(k) - 1, k+1) - h_i(Q_{i+1}^p(k) - 2, k+1) = h_i(Q_{i+1}^p(k) - 1, k) - h_i(Q_{i+1}^p(k) - 2, k)$, which is greater than $h_i(Q_{i+1}^p(k), k) - h_i(Q_{i+1}^p(k) - 1, k) > p_i^s$.

For the case of one more delivery in case $k+1$, $h_i(Q_{i+1}^p(k) - 1, k+1) - h_i(Q_{i+1}^p(k) - 2, k+1) = h_i(Q_{i+1}^p(k), k) - h_i(Q_{i+1}^p(k) - 1, k) \geq p_i^s$.

Therefore, we can conclude that $h_i(Q^p_{i+1}(k)-1,k+1)-h_i(Q^p_{i+1}(k)-2,k+1) \geq p^s_i$ and hence, $Q^p_{i+1}(k+1) \in \{Q^p_{i+1}(k)-1, Q^p_{i+1}(k)\}$.

B.9 Proof of Theorem 4.6

Given $(Q^p_i, Q^d_i, r_i, s_i, d_i, e_i)$, we slightly change the notation by replacing $\pi_i(Q^p_i, Q^d_i, r_i, s_i, d_i, e_i)$ into π^k_i if the remaining number of empty containers is k. Then we have

$$\pi^k_i = (p^r_i - p^s_i)r_i + (p^d_i - p^e_i)d_i + p^s_i Q^p_i + (p^s_i + p^e_i)Q^d_i - p^s_i Q^p_{i+1} - (p^s_i + p^e_i)k + h_i(Q^p_{i+1}, k).$$

Thereafter,

$$\pi^k_i - \pi^{k-1}_i = -(p^s_i + p^e_i) + p^s_i Q^p_{i+1}(k) + p^s_i Q^p_{i+1}(k-1) + h_i(Q^p_{i+1}(k), k) - h_i(Q^p_{i+1}(k-1), k-1).$$

Let $Q^{p*}_{i+1}(k)$ denote the optimal remaining available space by solving (4.9). There are two situations where $Q^{p*}_{i+1}(k) = Q^{p*}_{i+1}(k-1)$ or $Q^{p*}_{i+1}(k) = Q^{p*}_{i+1}(k-1) - 1$.

Consider $Q^{p*}_{i+1}(k) = Q^{p*}_{i+1}(k-1)$ first.

If both case k and case $k-1$ could achieve $Q^{p*}_{i+1}(k)$, then

$$\pi^k_i - \pi^{k-1}_i = (-p^s_i + p^e_i) + h_i(Q^{p*}_{i+1}(k), k) - h_i(Q^{p*}_{i+1}(k), k-1)$$

Due to the optimality condition, $h_i(Q^{p*}_{i+1}(k), k) \geq p^s_i + h_i(Q^{p*}_{i+1}(k)-1, k)$. Hence,

$$\pi^k_i - \pi^{k-1}_i \geq -p^e_i + h_i(Q^{p*}_{i+1}(k)-1, k) - h_i(Q^{p*}_{i+1}(k), k-1).$$

When $k < Q^{d*}_{i+1}$, we can conclude $-p^e_i + h_i(Q^{p*}_{i+1}(k)-1, k) - h_i(Q^{p*}_{i+1}(k), k-1) \geq 0$ from Lemma 4.3 and thus $\pi^k_i - \pi^{k-1}_i > 0$.

When $k > Q^{d*}_{i+1}$, with the optimality condition, we have $h_i(Q^{p*}_{i+1}(k), k-1) \geq p^s_i + h_i(Q^{p*}_{i+1}(k)+1, k-1)$. Therefore,

$$pi^k_i - \pi^{k-1}_i \leq -p^e_i + h_i(Q^{p*}_{i+1}(k), k) - h_i(Q^{p*}_{i+1}(k)+1, k-1).$$

As $-p^e_i + h_i(Q^{p*}_{i+1}(k), k) - h_i(Q^{p*}_{i+1}(k)+1, k-1) \leq 0$, we have $pi^k_i - \pi^{k-1}_i \leq 0$.

Otherwise, the feeder vessel will collect the same number of laden containers, which means $Q^p_{i+1}(k-1) = Q^p_{i+1}(k) + 1$. So we have

$$\pi^k_i - \pi^{k-1}_i = -p^e_i + h_i(Q^p_{i+1}(k), k) - h_i(Q^p_{i+1}(k)+1, k-1).$$

From Lemma 4.3, we will have $\pi_i^k - \pi_i^{k-1} \begin{cases} \geq 0, & \text{if } k < Q_{i+1}^{d*} \\ \leq 0, & \text{if } k > Q_{i+1}^{d*} \end{cases}$.

Now, consider $Q_{i+1}^{p*}(k) = Q_{i+1}^{p*}(k-1) - 1$. At this time, the decisions to collect laden containers in case k and case $k-1$ will always be same. It is same as the above case and thus the result holds.

B.10 Proof of Theorem 4.7

We simply use $\pi_i(Q_i^p, Q_i^d)$ instead of $\pi_i(Q_i^p, Q_i^d, r_i, s_i, d_i, e_i)$ since the result need hold for any r_i, s_i, d_i, e_i. Then to prove $\pi_i(Q_i^p, Q_i^d)$ is submodular on (Q_i^p, Q_i^d), we need to prove that

$$\pi_i(Q_i^p + 1, Q_i^d) - \pi_i(Q_i^p, Q_i^d) \geq \pi_i(Q_i^p + 1, Q_i^d + 1) - \pi_i(Q_i^p, Q_i^d + 1).$$

It is easy to check that $\pi_n(Q_n^p, Q_n^d)$ is submodular function on (Q_n^p, Q_n^d) for any r_n, s_n, d_n, e_n.

Now assume $\pi_{i+1}(Q_{i+1}^p, Q_{i+1}^d)$ is submodular function on (Q_{i+1}^p, Q_{i+1}^d) for any $r_{i+1}, s_{i+1}, d_{i+1}, e_{i+1}$. Hence, $h_i(Q_{i+1}^p, Q_{i+1}^d)$ is submodular function on (Q_{i+1}^p, Q_{i+1}^d).

Let x_A and x_B denote the optimal amount of empty containers to be delivered if the capacity is (Q_i^p+1, Q_i^d) and (Q_i^p, Q_i^d). We can easily conclude that $x_A = x_B$ or $x_A = x_B - 1$.

The left hand side is then equivalent to

$$p_i^e x_A + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) - p_i^e x_B - \pi_i(Q_i^p + x_B, Q_i^d - x_B).$$

If the capacity is $(Q_i^p + 1, Q_i^d + 1)$, the optimal amount of empty container will be either x_A or $x_A + 1$. If the capacity is $(Q_i^p, Q_i^d + 1)$, the optimal amount of empty container will be either x_B or $x_B + 1$. The right hand can thus be divided into four scenarios.

For the case of x_A and x_B occur, the right hand side will be

$$p_i^e x_A + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - p_i^e x_B - \pi_i(Q_i^p + x_B, Q_i^d - x_B + 1).$$

For the case of $x_A + 1$ and $x_B + 1$ occur, the right hand side will be

$$p_i^e(x_A + 1) + \pi_i(Q_i^p + x_A + 2, Q_i^d - x_A) - p_i^e(x_B + 1) - \pi_i(Q_i^p + x_B + 1, Q_i^d - x_B).$$

For the case of $x_A + 1$ and x_B occur, the right hand side will be

$$p_i^e(x_A + 1) + \pi_i(Q_i^p + x_A + 2, Q_i^d - x_A) - p_i^e x_B - \pi_i(Q_i^p + x_B, Q_i^d - x_B + 1).$$

For the case of x_A and x_B+1, the right hand hand will be

$$p_i^e x_A + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - p_i^e(x_B+1) - \pi_i(Q_i^p + x_B + 1, Q_i^d - x_B).$$

We discuss the above four cases one by one. We start with $X_A = X_B$. At this time, $\Pr(x_A+1,\, x_B \text{ occurs}) = 0$. Therefore, we do not need to consider the third case here.

The left hand side is then

$$\pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) - \pi_i(Q_i^p + x_A, Q_i^d - x_A).$$

The right hand side in the first case will be $p_i^e x_A + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - p_i^e x_B - \pi_i(Q_i^p + x_B, Q_i^d - x_B + 1) = \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - \pi_i(Q_i^p + x_A, Q_i^d - x_A + 1)$.

Let Q_{i+1}^{p*} and $\hat{Q}_{i+1}^{p*}$ denote the optimal remaining capacity for the collection when the remaining capacity for the delivery is $Q_{i+1}^d - x_A$ and $Q_{i+1}^d - x_A + 1$, respectively. Here, we know that $\hat{Q}_{i+1}^{p*} = Q_{i+1}^{p*}$ or $Q_{i+1}^{p*} - 1$.

If $\hat{Q}_{i+1}^{p*} = Q_{i+1}^{p*}$, the number of laden containers the vessel will collect will be the same for the left hand side and the right hand side. And we can conclude that the right hand side is less than the left hand side due to the concavity.

Next, consider if $\hat{Q}_{i+1}^{p*} = Q_{i+1}^{p*} - 1$. If the left hand side and right hand side both achieve the optimal remaining capacity for the collection, then it will be equivalent to $p_i^s = p_i^s$. If no case achieves the optimal remaining capacity, it means the decision will be on r_i or $r_i + s_i$ and we and thus conclude the the left hand side is greater than the right hand side due to the submodularity. If $(Q_i^p + x_A, Q_i^d - x_A + 1)$ can achieve the optimal remaining capacity by collecting all laden containers, then only case $(Q_i^p + x_A + 1, Q_i^d - x_A + 1)$ can not achieve the optimal remaining capacity. Therefore, the left hand side is p_i^s and the right hand side is $h_i(Q_{i+1}^{p*}, Q_i^d - x_A + 1) - h_i(Q_{i+1}^{p*}, Q_i^d - x_A + 1)$. The left hand side is greater than the right hand side due to the optimality condition that $Q_{i+1}^{p*} - 1$ is the optimal solution given remaining capacity for delivery being $Q_i^d - x_A + 1$. Therefore, the left hand side is always greater than the right hand side.

The second case is $p_i^e(x_A+1) + \pi_i(Q_i^p + x_A + 2, Q_i^d - x_A) - p_i^e(x_B+1) - \pi_i(Q_i^p + x_B + 1, Q_i^d - x_B) = \pi_i(Q_i^p + x_A + 2, Q_i^d - x_A) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A)$, which is less than the left hand side due to the concavity.

The fourth case is $p_i^e x_A + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - p_i^e(x_B + 1) - \pi_i(Q_i^p + x_B + 1, Q_i^d - x_B) = -p_i^e + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A)$. Since $x_A + 1$ is the optimal solution if the capacity is $(Q_i^p, Q_i^d + 1)$, $p_i^e + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) - \pi_i(Q_i^p + x_A, Q_i^d - x_A + 1) \geq 0$, which implies that $\pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - \pi_i(Q_i^p + x_A, Q_i^d - x_A + 1) \geq -p_i^e + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A)$. Now that $\pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - \pi_i(Q_i^p + x_A, Q_i^d - x_A + 1)$ is less than the left hand side, $p_i^e x_A + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - p_i^e(x_B + 1) - \pi_i(Q_i^p + x_B + 1, Q_i^d - x_B)$ is less than the left hand side.

Applying same method if $x_A = x_B - 1$, we will still have the left hand side is greater than the right hand side.

The left hand side is equivalent to

$$-p_i^e + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A - 1).$$

The first case will be $p_i^e x_A + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - p_i^e x_B - \pi_i(Q_i^p + x_B, Q_i^d - x_B + 1) = -p_i^e + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A + 1) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A)$. We can conclude that the left hand side is greater than the right hand side due to the concavity.

The second case is then $-p_i^e + \pi_i(Q_i^p + x_A + 2, Q_i^d - x_A) - \pi_i(Q_i^p + x_A + 2, Q_i^d - x_A - 1)$. We can conclude the left hand side is greater same as the first case when $x_A = x_B$.

The third case is then $\pi_i(Q_i^p + x_A + 2, Q_i^d - x_A) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A)$. Due to the concavity, $\pi_i(Q_i^p + x_A + 2, Q_i^d - x_A) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) \leq \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) - \pi_i(Q_i^p + x_A, Q_i^d - x_A)$. Since x_A is the optimal solution if the capacity is (Q_i^p, Q_i^d), we will have $p_i^e + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A - 1) \leq \pi_i(Q_i^p + x_A, Q_i^d - x_A)$, which implies that $-p_i^e + \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) - \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A - 1) \geq \pi_i(Q_i^p + x_A + 1, Q_i^d - x_A) - \pi_i(Q_i^p + x_A, Q_i^d - x_A)$. Therefore, the left hand side is greater than the right hand side.

The probability of the fourth case is 0.

Therefore, the left hand side is greater than the right hand side.

Above all, the right hand side is less than the left hand side and $\pi_i(Q_i^p, Q_i^d)$ is submodular function on (Q_i^p, Q_i^d).